The Lambda-calculus with multiplicities

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(rapport préliminaire)

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Abstract.

We introduce a refinement of the λ-calculus, where the argument of a function is a bag of resources, that is a multiset of terms, whose multiplicities indicate how many copies of them are available. We show that this "λ-calculus with multiplicities" has a natural functionality theory, similar to Coppo and Dezani's intersection type discipline. In our functionality theory the conjunction is managed in a "multiplicative" manner, according to Girard's terminology. We show that this provides an adequate interpretation of the calculus, by establishing that a term is convergent if and only if it has a non-trivial functional character.

Résumé.

On introduit un raffinement du λ-calcul dans lequel l'argument d'une fonction est un paquet de ressources, c'est à dire un multi-ensemble de termes, dont les multiplicités indiquent combien de copies en sont disponibles. Nous montrons que ce "λ-calcul avec multiplicités" a une théorie de la fonctionnalité naturelle, similaire à la théorie des types avec intersection de Coppo et Dezani. Dans notre théorie de la fonctionnalité la conjonction est traitée de manière "multiplicative", selon la terminologie de Girard. Nous montrons que ceci fournit une interprétation adéquate de notre calcul, en établissant qu'un terme est convergent si et seulement si il a un caractère fonctionnel non trivial.

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We introduce a refinement of the λ-calculus, where the argument of a function is a bag of resources, that is a multiset of terms, whose multiplicities indicate how many copies of them are available. We show that this "λ-calculus with multiplicities" has a natural functionality theory, similar to Coppo and Dezani's intersection type discipline. In our functionality theory the conjunction is managed in a "multiplicative" manner, according to Girard's terminology. We show that this provides an adequate interpretation of the calculus, by establishing that a term is convergent if and only if it has a non-trivial functional character.

1. Introduction.

Following Girard, the λ-calculus application could be written $M(!N)$, or MN^∞, as we shall do, instead of MN, to emphasize the fact that the argument N is actually *infinitely available* for the function M. Indeed, in the term resulting from a β-reduction $(\lambda x.M)N \to M[N/x]$, the argument N is copied as many times as needed, that is, as much as there are free occurrences of x in M. Therefore, one could also use an alternative notation for substitution, like $M\langle N^\infty/x\rangle$.

This immediately suggests to investigate a more general situation where we write MN^m and $M\langle N^m/x\rangle$, meaning that the argument is of possibly limited availability, that is $m \in \mathbb{N} \cup \{\infty\}$. The β-reduction remains the same, that is $(\lambda x.M)N^m \to M\langle N^m/x\rangle$, but it is perhaps less obvious to see how to perform the substitution. A natural way is to fetch a sample of the resource whenever it is needed, that is when the variable occurs in the head position. To formalize this, it is convenient to use explicit substitutions (or "closures", in Landin's terminology), as proposed by Curien *et al.*

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[1,10], turning the meta-operation of substitution into a computational device. Then we may define the evaluation process in such a way that, provided x is not free in M:

$$(xN_1^{m_1}\cdots N_k^{m_k})\langle M^m/x\rangle \to (MN_1^{m_1}\cdots N_k^{m_k})\langle M^{m-1}/x\rangle$$

Clearly what is new is the possibility of *deadlock*: if there is no resource available for x, like in $(xN_1^{m_1}\cdots N_k^{m_k})\langle M^0/x\rangle$, then no reduction is possible (here we follow the *lazy* strategy of Abramsky and Ong [2]; then we do not evaluate the arguments). However, we do not wish to regard this normal form as a meaningful value. Then introducing finite multiplicities provides us with more discriminating power. For instance, we distinguish the two λ-terms xx and $x(\lambda y.xy)$ – that is more precisely (xx^∞) and $(x(\lambda y.(xy^\infty))^\infty)$ –, since in the context $[\cdot]\langle \mathbf{I}^1/x\rangle$, where $\mathbf{I} = \lambda z.z$ is the identity, the first one reduces to a deadlock $x\langle x^\infty/z\rangle\langle \mathbf{I}^0/x\rangle$, while the second has a value, namely the closure $(\lambda y.(xy^\infty))\langle(\lambda y.(xy^\infty))^\infty/z\rangle\langle \mathbf{I}^0/x\rangle$.

This gain in discriminating power was our original motivation for introducing the λ-calculus with multiplicities, which emerged from the study of the encoding of the lazy λ-calculus into the π-calculus given by Milner in [17]. He showed that his encoding is adequate, that is, two λ-terms that are separated by λ-calculus contexts are also separated in the π-calculus, but not fully abstract (or not fully adequate), that is, the π-calculus is strictly more discriminating. There are several reasons for this. One reason is that the lazy λ-calculus is too weak to define some strict functions, like the "convergence testing" combinator, see [2]. Milner showed that the π-calculus has some convergence testing ability, thus allowing for instance the two terms of Ong's example to be discriminated (see [2,17]). Let us note that the same context as above, $[\cdot]\langle \mathbf{I}^1/x\rangle$, may be used for this particular purpose.

Another weakness of the lazy λ-calculus is that it lacks means for defining parallel functions. That is, there are terms that are not distinguished by λ-calculus contexts (in the lazy regime), but can be differentiated by means of parallel functions (see [2]). In [5,6] we showed that functional parallelism is essentially the "join" operation, which may be implemented either using a non-deterministic choice or as a confluent parallel composition. Obviously, the π-calculus offers such non-deterministic and parallel facilities, thus providing another source of extra discriminating power w.r.t. the lazy λ-calculus. We shall return to this point later.

Finally, there are some quite "robust" λ-calculus equalities – like the afore-mentioned one, $xx = x(\lambda y.xy)$, which holds in any model satisfying a weak form of extensionality, namely $M \neq \Omega \Rightarrow M = \lambda x(Mx)$, for x not free in M – which can be broken using π-calculus features. Indeed, this is because the π-calculus provides means, namely parallel composition and replication (or "bang"), for controlling the number of copies of an agent: a resource with finite multiplicity is simply the parallel composition of copies of the resource, as many times as we wish, i.e. $M^m = (M|\cdots|M)$, m times, while M^∞ is the replication of M, also denoted $!M$, which may be regarded as an infinite parallel composition of copies of M. Therefore, to give our calculus its full generality, we shall not confine ourselves to arguments of the form N^m, but we shall allow *bags of resources* as arguments. We write a bag, that is more formally a multiset of terms, as a parallel composition of terms with multiplicities:

$$\begin{aligned} P &= (M_1^{m_1} \mid \cdots \mid M_k^{m_k} \mid N_1^\infty \mid \cdots \mid N_n^\infty) \\ &= (\underbrace{(M_1 \mid \cdots \mid M_1)}_{m_1 \text{ times}} \mid \cdots \mid \underbrace{(M_k \mid \cdots \mid M_k)}_{m_k \text{ times}} \mid N_1^\infty \mid \cdots \mid N_n^\infty) \end{aligned}$$

The parallel composition is intended to be commutative and associative, with a neutral element **1**, denoting the empty multiset. Now we may give the syntax of our *λ-calculus with multiplicities*:

$$\begin{array}{lll} M ::= x \mid \lambda x.M \mid (MP) \mid (M\langle P/x\rangle) & & (\textit{terms, or "programs"}) \\ P ::= \mathbf{1} \mid M \mid (P \mid P) \mid M^{\infty} & & (\textit{bags of terms}) \end{array}$$

The rules for (lazy) evaluation are given in the next technical section. In particular, the "fetch" operation implementing substitution now becomes:

$$(xN_1^{m_1} \cdots N_k^{m_k})\langle (M \mid P)/x\rangle \to (MN_1^{m_1} \cdots N_k^{m_k})\langle P/x\rangle$$

What is new now is that the evaluation is *non-deterministic*, since one may fetch a sample M_i of any of the resources contained in a bag $P = (M_1^{m_1} \mid \cdots \mid M_k^{m_k})$, provided $m_i > 0$ – recall that parallel composition is commutative and associative. This is best exemplified by defining a *non-deterministic (internal) choice*:

$$(M \oplus N) =_{\text{def}} x\langle (M \mid N)/x\rangle$$

Then one can see that, up to some garbage collection:

$$(M \oplus N) \to M \qquad \text{and} \qquad (M \oplus N) \to N$$

The non-deterministic choice contruct may be used to build parallel functions, as shown in [5]. Sangiorgi showed in [19,20] that in a sense internal choice is exactly what makes the π-calculus more powerful than the lazy λ-calculus: he established that the encoding of the latter, extended with internal choice, into the π-calculus is fully abstract. Then, as he pointed out to me, the encoding of the λ-calculus with multiplicities should enjoy the same property. His result was established considering a kind of *bisimulation* as the semantical equality for extended λ-terms. Here we shall use a more standard semantics, known as Morris' extensional operational semantics, based on the discriminating power offered by the contexts of the calculus. More precisely, we define the *testing preorder*, as follows:

$$M \sqsubseteq N \;\Leftrightarrow_{\text{def}}\; \forall C \text{ closing } M \text{ and } N.\;\; C[M]\Downarrow \;\Rightarrow\; C[N]\Downarrow$$

where $M\Downarrow$ means that this term has a value.

Now let us turn our attention to the question of the "abstract" semantics – abstract in the sense of not mentioning the evaluation process. Incidentally we note that, since we intuitively presented our λ-calculus with multiplicities in terms of resource availability, one would expect some connection with Girard's Linear Logic [11], and more specifically with the "Bounded Linear Logic" of Girard *et al.* [13]. We shall leave this matter open for further investigations. However, in this preliminary report we shall use some of the ideas of Linear Logic, for the purpose of giving a semantics for our calculus. More precisely, we introduce a refinement of the "intersection type discipline" of Coppo *et al.*, and prove refined versions of results of [8,9].

The intersection type discipline was introduced by Coppo and Dezani in [7,8], and independently by Sallé [18], to the purpose of assigning meaningful functional characters to terms that do not have any type in Curry's functionality theory. Namely, a λ-term may be typed in a non-trivial way in this type discipline if and only it has a head normal form. A crucial fact for this result is the "subject expansion" property, which states that if a λ-term R reduces to R', and R' has type τ, then R has type τ too. Typically, this holds for $R = (\lambda x.M)N$ and $R' = M[N/x]$. Then one shows that from a typing of $M[^N/_x]$ one can build a typing of M under the assumption that x has a type that N possesses. This is something like a converse of the cut rule, which we called the "paste property" in [6].

To achieve this paste property, the intersection type discipline uses as type constructors, besides Curry's arrow $\sigma \rightarrow \tau$, a constant ω and a conjunction $\sigma \wedge \tau$. The constant ω is the universal type – the "truth" –, which may be assigned to any term. Then ω may be used as a type for x in a typing of M whenever x does not occur in M. On the other hand, whenever x occurs more than once in M, and $\tau_1, \ldots, \tau_n$ are the types assigned to the various occurrences of N in a typing of $M[^N/_x]$, one uses the conjunction, to build a typing of M under the assumption that x has the type $\tau_1 \wedge \cdots \wedge \tau_n$. As it was originally done in [8,9], one may restrict the use of conjunction (except for the empty one, that is ω) to the left of the arrow without affecting the main results.

As one can see, the use of conjunction is related to the implicit multiplicity of the arguments in the λ-calculus(1). Then it will not be surprising that, to establish a functionality theory for our λ-calculus with multiplicities enjoying similar properties, we shall also use a conjunction for accumulating the information about the arguments – in our case the bags of resources. However, since a resource is used at most once in our calculus, we shall use a specific management of the conjunction, inspired by Girard's notion of a *multiplicative* connective [11]. We must immediately point out that we do *not* use our calculus to develop a proof theory (i.e. a Curry-Howard correspondence) for any fragment of Linear Logic. As a matter of fact, our "logic" will be quite different from Linear Logic: our conjunction is neither the additive nor the multiplicative one(2). This is because we shall use an unrestricted form of weakening(3), corresponding to the fact that a resource may be useless, but disallow the use of contraction, since a resource cannot be used twice. The only difference with Coppo and Dezani's conjunction is that ours is not idempotent (except for the empty one). Therefore, to avoid any confusion, we use the following concrete syntax for functional characters:

$$\phi ::= \omega \mid (\pi \rightarrow \phi)$$
$$\pi ::= \phi \mid (\pi \times \pi)$$

where $\times$ is our conjunction. Apart from the use of this last symbol, our set of functional characters is the same as the one of [8,9].

Then we shall introduce a functionality theory, that is a kind of typing system allowing us to infer sequents of the form $x_1 : \pi_1, \ldots, x_k : \pi_k \vdash M : \phi$, which may be read: "under the hypothesis that each free occurrence of x_i in M represents exactly one of the resources of a bag satisfying π_i, one may conclude that M has the character ϕ". Next we define the interpretation of a term M as the set $\mathcal{F}[\![M]\!]$ of pairs (Γ, ϕ) such that $\Gamma \vdash M : \phi$. Our main result is that this interpretation is *adequate*, that is:

$$\mathcal{F}[\![N]\!] \subseteq \mathcal{F}[\![M]\!] \;\Rightarrow\; N \sqsubseteq M$$

The paper is organized as follows: after having settled some syntactic matters, we define the lazy evaluation process. Then we introduce the testing preorder, and establish some of its properties, using a "context lemma". Next we show that our λ-calculus with multiplicities is a strict refinement of the lazy λ-calculus, in the sense that there is an adequate but not fully abstract translation from the latter to the former. Actually the translation goes into a sub-calculus, where we only use MN^m and $M\langle N^m/x\rangle$, but no explicit parallel composition. Then we introduce the functionality system,

(1) another way of dealing with implicit multiplicities is to use polymorphic types.
(2) it could be $(\mathbb{1} \,\&\, A) \otimes (\mathbb{1} \,\&\, B)$, see the Approximation Theorem of [11].
(3) this also explains why we do not need to distinguish two kinds of truth.

and undertake the proof of the afore-mentioned adequation theorem. A first step is to show the "subject expansion" property. We establish this property for a notion of reduction that contains the evaluation process, and allows computing in any context. Finally we introduce a notion of realizability of formulae, and conclude the proof of the theorem by showing that the typing system is sound with respect to realizability.

2. Syntax and Evaluation.

2.1 Syntax.

We recall that the grammar of our calculus is, as given in the introduction:

$$\begin{aligned} M &::= x \mid \lambda x.M \mid (MP) \mid (M\langle P/x\rangle) \\ P &::= \mathbf{1} \mid M \mid (P \mid P) \mid M^\infty \end{aligned}$$

We denote by Λ^m the set of terms given by the first clause of the grammar, and by Π the set of "bags of resources", that is the terms given by the second clause of the grammar. We use S, $T\ldots$ to stand for terms of any kind. We adopt the usual conventions of the λ-calculus; in particular $MP_1 \cdots P_k$ stands for $(\cdots(MP_1)\cdots P_k)$. We shall also use a notation, namely Σ, for the set of substitution items, that is the terms of the form $\langle P/x\rangle$ for $P \in \Pi$. Then $MQ_1 \cdots Q_k$ is an abbreviation for $(\cdots(MQ_1)\cdots Q_k)$ where the Q_i's belong to $\Pi \cup \Sigma$. The variable x is *bound* in M by the construct $M\langle P/x\rangle$. We define the α-conversion accordingly; the definition is given in the appendix. We denote by $\mathsf{fv}(M)$ the set of free variables of M (see the appendix).

We shall see that the lazy λ-calculus is embedded in our λ-calculus with multiplicities, Namely, the usual application is recovered as MN^∞, and the ordinary substitution is $M\langle N^\infty/x\rangle$. Then we shall keep the standard notation for the usual combinators, written with explicit multiplicities. For instance, Δ denotes $\lambda x(xx^\infty)$ and $\Omega = (\Delta\Delta^\infty)$.

As we said in the introduction, the parallel composition is intended to be commutative and associative, with $\mathbf{1}$ as a neutral element. Moreover, M^∞ is an infinite parallel composition of copies of M. Then the "bags of resources" are really elements of $\Pi/\equiv$, where $\equiv$ is the *structural equivalence* (see [17]), that is the least congruence over Π satisfying:

$$\begin{aligned} (P \mid \mathbf{1}) &\equiv P \\ (P \mid Q) &\equiv (Q \mid P) \\ (P \mid (Q \mid R)) &\equiv ((P \mid Q) \mid R) \\ M^\infty &\equiv (M \mid M^\infty) \end{aligned}$$

We shall use $(P_1 \mid \cdots \mid P_n \mid P_{n+1})$ as an abbreviation for $(P_1 \mid (\cdots \mid (P_n \mid P_{n+1})\cdots))$. One can see that the bags of resources are "finite" multisets of terms of Λ^m. More precisely, let us define the terms with explicit finite multiplicity, as follows:

$$\begin{aligned} M^0 &= \mathbf{1} \\ M^{m+1} &= (M \mid M^m) \end{aligned}$$

Then for any $P \in \Pi$:

$$P \equiv (M_1^{m_1} \mid \cdots \mid M_k^{m_k})$$

where the multiplicities $m_1, \ldots, m_k$ are possibly infinite, i.e. $m_i \in \mathbb{N} \cup \{\infty\}$. The structural equivalence is extended to terms of Λ^m as the least congruence, still denoted $\equiv$, such that

$$P \equiv Q \;\Rightarrow\; MP \equiv MQ \quad \& \quad M\langle P/x\rangle \equiv M\langle Q/x\rangle$$

2.2 Evaluation.

As we said in the introduction, the evaluation mechanism we define now follows a *lazy* strategy, that is we do not evaluate the body M of an abstraction $\lambda x.M$, and we do not evaluate the arguments, that is more accurately the bag of resources P in MP and $M\langle P/x\rangle$. The one-step evaluation relation is denoted $M \rightarrow M'$. We shall use implicitly the rule that any two α-convertible terms have the same reductions:

$$\frac{M \rightarrow M'}{N \rightarrow M'}\ M =_\alpha N$$

To introduce the evaluation mechanism, we observe that any term M may be written in a unique way as $AQ_1 \cdots Q_k$ where the Q_i's belong to $\Pi \cup \Sigma$, and A is either a variable, or an abstraction $\lambda x.N$. We may call A the *head subterm* of M. Indeed, we could have written the grammar for terms as follows:

$$\begin{aligned} M &::= A \mid (MQ) \\ A &::= x \mid \lambda x.M \\ Q &::= P \mid \langle P/x\rangle \end{aligned}$$

where P stands for any term of Π. We shall also use a notion of *value*, which is any functional closure. That is, the values are the terms of Λ^m given by the grammar:

$$V ::= \lambda x.M \mid (V\langle P/x\rangle)$$

We use $V, W \ldots$ to range over values. Now we introduce the computation rules. The first two state that a reduction may be performed in the context of a list of arguments or substitutions:

$$\text{E1:}\ \frac{M \rightarrow M'}{MP \rightarrow M'P} \qquad \text{E2:}\ \frac{M \rightarrow M'}{M\langle P/x\rangle \rightarrow M'\langle P/x\rangle}$$

Then the actual computation depends on the form of the head subterm A of $M = AQ_1 \cdots Q_k$. When A is an abstraction, we look for the first Q_i in the list, if any, which is an argument, that is a term P of Π to which the closure $AQ_1 \cdots Q_{i-1}$ is applied. Then, possibly using the context rules E1 and E2, we perform a β-reduction of the form:

$$(\lambda x.M)\langle P_1/x_1\rangle \cdots \langle P_{i-1}/x_{i-1}\rangle PQ_{i+1} \cdots Q_k \rightarrow M\langle P/x\rangle\langle P_1/x_1\rangle \cdots \langle P_{i-1}/x_{i-1}\rangle Q_{i+1} \cdots Q_k$$

provided the x_j's are not free in P. This is formalized by the following two rules:

$$\text{E3:}\ (\lambda x.M)P \rightarrow M\langle P/x\rangle \qquad \text{E4:}\ \frac{(VP) \rightarrow M}{(V\langle R/x\rangle)P \rightarrow M\langle R/x\rangle}\ x \notin \mathsf{fv}(P)$$

When the head subterm is a variable x, one looks for the first substitution $\langle P/x\rangle$ for it, if any, in the list $Q_1 \cdots Q_k$. Then one fetches a resource out of P, that is any term N of Λ^m such that $P \equiv (N \mid R)$, and leaves the rest R for future use. To state the "fetch" rule E5, we introduce an auxiliary relation $M\langle N/x\rangle \mapsto M'$, intended to formalize the replacement of the head variable x of M by N, that is, roughly, $M = xQ_1 \cdots Q_k$ and $M' = NQ_1 \cdots Q_k$, where $Q_i \in \Pi \cup \Sigma$. The rules are:

$$\text{S1:}\ x\langle M/x\rangle \mapsto M \qquad \text{S2:}\ \frac{M\langle N/x\rangle \mapsto M'}{(MP)\langle N/x\rangle \mapsto M'P}$$

$$\text{S3:}\quad \frac{M\langle N/x\rangle \mapsto M'}{(M\langle P/z\rangle)\langle N/x\rangle \mapsto M'\langle P/z\rangle}\quad z \neq x \;\&\; z \notin \mathsf{fv}(N)$$

Now we can give the "fetch" rule, which is the last rule of evaluation:

$$\text{E5:}\quad \frac{M\langle N/x\rangle \mapsto M'}{M\langle P/x\rangle \to M'\langle R/x\rangle}\quad P \equiv (N \mid R),\ x \notin \mathsf{fv}(N)$$

Note that in this rule the resource which is fetched, that is N, is a term of Λ^m, and not a term of Π.

We write $M\Downarrow$ whenever M has a value, that is $\exists V.\ M \xrightarrow{*} V$. If this is not the case, we write $M\Uparrow$. For instance $\Omega\Uparrow$. Note that $MP\Downarrow \;\Rightarrow\; M\Downarrow$ and $M\Downarrow \;\Rightarrow\; M\langle P/x\rangle\Downarrow$. We could have considered a "garbage collection" rule:

$$\frac{}{M\langle P/x\rangle \to M}\quad x \notin \mathsf{fv}(M)$$

We shall see that this rule is semantically valid. Note also that since $M\langle N/x\rangle \mapsto M'$ means that N replaces the variable x when it is needed, that is when x is at the head position in M, one could think of adopting a "call-by-need" strategy. That is, one could require that this replacement takes place only when N has been evaluated. However, this does not change the convergence predicate $M\Downarrow$ since in $NQ_1\cdots Q_k$, the subterm N has to be evaluated anyway. Now let us see some examples of evaluation. Since $M \equiv (M \mid \mathbf{1})$ we have, using S1 and E5:

$$x\langle M/x\rangle \to M\langle \mathbf{1}/z\rangle$$

where z is not free in M. Moreover, up to some garbage collection, $M\langle \mathbf{1}/z\rangle$ is identical to M. To see the use of infinitely available resources M^∞, note that since $M^\infty \equiv (M \mid M^\infty)$, we have:

$$(xQ_1\cdots Q_k)\langle M^\infty/x\rangle \to (MQ_1\cdots Q_k)\langle M^\infty/x\rangle$$

provided x is not free in M (and the additional constraints imposed by S3 are fulfilled). Let us see another example, showing how the evaluation in the usual lazy λ-calculus is mimicked in the λ-calculus with multiplicities (this will be investigated in a more formal way below). Recall that the application of the usual λ-calculus is modelled as (MN^∞), while $M\langle N^\infty/x\rangle$ represents the usual substitution. Let $\mathsf{I} = \lambda z.z$ and $\Delta = \lambda x(xx^\infty)$. Then for instance we have:

$$\begin{aligned}
(\Delta\,\mathsf{I}^\infty) &\to (xx^\infty)\langle \mathsf{I}^\infty/x\rangle && \text{(E3)}\\
&\to (\mathsf{I}\,x^\infty)\langle \mathsf{I}^\infty/x\rangle && \text{(S1, S2, E5)}\\
&\to z\langle x^\infty/z\rangle\langle \mathsf{I}^\infty/x\rangle && \text{(E3, E2)}\\
&\to x\langle x^\infty/z\rangle\langle \mathsf{I}^\infty/x\rangle && \text{(S1, E5, E2)}\\
&\to \mathsf{I}\,\langle x^\infty/z\rangle\langle \mathsf{I}^\infty/x\rangle && \text{(S1, S3, E5)}
\end{aligned}$$

Using a garbage collection rule, the last term of this sequence of evaluations would be transformed into I.

We pointed out in the introduction that what makes a real difference with the usual λ-calculus is that the evaluation mechanism is non-deterministic, and may introduce deadlocks. We defined the non-deterministic choice as follows:

$$(M \oplus N) =_{\mathsf{def}} x\langle (M \mid N)/x\rangle$$

Since $(M \mid N) \equiv (N \mid M)$, we have both $(M \oplus N) \rightarrow M\langle N/z\rangle$ and $(M \oplus N) \rightarrow N\langle M/z\rangle$, with $z \notin \mathsf{fv}(M) \cup \mathsf{fv}(N)$. Moreover, the two terms $M\langle N/z\rangle$ and $N\langle M/z\rangle$ may be regarded as identical to M and N respectively, up to some garbage collection. Therefore one has - again up to some identifications:

$$(M \oplus N) \rightarrow M \qquad \text{and} \qquad (M \oplus N) \rightarrow N$$

Clearly, the commutativity of parallel composition is the only source of non-determinism of the evaluation mechanism. However, the evaluation is still deterministic if for instance we restrict the use of parallel composition to build terms with multiplicities, that is if we consider the terms of the set $\Lambda^\star$, given by the grammar:

$$M ::= x \mid \lambda x.M \mid (MM^m) \mid (M\langle M^m/x\rangle)$$

where $m \in \mathbb{N} \cup \{\infty\}$. Indeed, it is easy to check that the following holds:

LEMMA 2.1. *For any $M \in \Lambda^\star$ we have: $M \rightarrow M' \Rightarrow \exists!\, N \in \Lambda^\star.\ M' \equiv N$.*

Note that for this $\Lambda^\star$ sub-calculus, one could optimize the evaluation process using a "call-by-need" mechanism, which is a refinement of the rule E5, given as follows:

$$\frac{N \xrightarrow{*} U\ ,\ M\langle U/x\rangle \mapsto M'}{M\langle N^{m+1}/x\rangle \rightarrow M'\langle U^m/x\rangle}\ x \notin \mathsf{fv}(U)$$

where U is either a value, or a term of the form $zQ_1 \cdots Q_k$, where no Q_i is a substitution for z (see also [22]).

Regarding the point of potential deadlocks, one can see that a term like $xQ_1 \cdots Q_k\langle \mathbf{1}/x\rangle$ has no evaluation if none of the Q_i's is a substitution for x, but this "normal form" is not regarded as a value, that is $xQ_1 \cdots Q_k\langle \mathbf{1}/x\rangle\Uparrow$. Then for instance if we consider $(\Delta\, \mathbf{I})$ instead of $(\Delta\, \mathbf{I}^\infty)$ (like in a previous example) we have:

$$\begin{aligned}(\Delta\, \mathbf{I}) &\rightarrow (xx^\infty)\langle \mathbf{I}/x\rangle\\ &\rightarrow (\mathbf{I}\, x^\infty)\langle \mathbf{1}/x\rangle\\ &\rightarrow z\langle x^\infty/z\rangle\langle \mathbf{1}/x\rangle\\ &\rightarrow x\langle x^\infty/z\rangle\langle \mathbf{1}/x\rangle\end{aligned}$$

This last term is identical, up to garbage collection, to $x\langle \mathbf{1}/x\rangle$, a deadlocked term.

2.3 The Testing Preorder.

Let us now define the operational semantics of the calculus, and give some of its basic properties. The operational semantics we adopt is the extensional preorder of Morris, which we call here the testing preorder. We call *test* any context C of the λ-calculus with multiplicities, that is any term built using the contructs of the calculus plus an additional constant $[]$, representing a "hole". As usual, we denote by $C[M]$ the term resulting from filling the hole in C by M (we say that C closes M if the free variables of M are bound by C). This also represents the testing of M by C. A success is reported whenever the evaluation of $C[M]$ terminates on a value, and a term is operationally better than another if it passes successfully more tests. The testing preorder is thus defined:

DEFINITION (TESTING).

$$M \sqsubseteq N \Leftrightarrow_{\mathrm{def}} \forall C \text{ closing } M \text{ and } N.\ \ C[M]\Downarrow \Rightarrow C[N]\Downarrow$$

Although the definition of the testing preorder is a "natural" one, it is not very convenient to prove any property - apart from the fact that the preorder is a precongruence! Then we shall use an alternative characterization of the testing preorder, showing that the tests may be restricted to "application to a sequence of arguments or substitutions". To this end, let us define the *applicative testing*, as follows:

$$M \sqsubseteq_{\mathcal{A}} N \Leftrightarrow_{\mathrm{def}} \forall k\, \forall Q_1, \ldots, Q_k \in \Pi \cup \Sigma\, (MQ_1 \cdots Q_k \Downarrow \;\Rightarrow\; NQ_1 \cdots Q_k \Downarrow)$$

where $MQ_1 \cdots Q_k$ and $NQ_1 \cdots Q_k$ are assumed to be closed. Then we have the standard result, known as the "context lemma" - or the property of "operational extensionality", see [2] for instance.

LEMMA (the CONTEXT LEMMA) 2.2.

$$M \sqsubseteq N \Leftrightarrow M \sqsubseteq_{\mathcal{A}} N$$

PROOF. The "$\Rightarrow$" direction is obvious. To prove the converse, we use Lévy's technique: if C is a context (closing M and N) such that $C[M] \xrightarrow{*} V$ for some value V, we show that $C[N]\Downarrow$ by induction on (l, h) (w.r.t. the lexicographic ordering), where l is the length of the evaluation sequence $C[M] \xrightarrow{*} V$, and h is number of occurrences of the hole $[]$ in C. We may write $C = C_0 C_1 \cdots C_n$ where C_0 is either the hole $[]$, or a variable x, or an abstraction context $\lambda x.B$, and the C_i's, for $i > 0$, are "argument contexts" (this notion should be clear). We examine the possible cases.

- $C_0 = []$. Let C' be the context $MC_1 \cdots C_n$, which has $h-1$ holes. Clearly C' closes both M and N. Since $C'[M] = C[M]$ we have $C'[M] \xrightarrow{l} V$, therefore by induction hypothesis we get $C'[N]\Downarrow$. Since $C'[N] = MC_1[N] \cdots C_n[N]$ and $M \sqsubseteq_{\mathcal{A}} N$ we conclude $NC_1[N] \cdots C_n[N]\Downarrow$, that is $C[N]\Downarrow$.
- $C_0 = x$. Here $l > 0$, and there exists i $(1 \leq i \leq n)$ such that $C_i = \langle D/x \rangle$, with $D[M] \equiv (T \mid R)$, where $T \in \Lambda^m$, $R \in \Pi$ and

$$C[M] \rightarrow TC_1[M] \cdots C_{i-1}[M]\langle R/x \rangle C_{i+1}[M] \cdots C_n[M] \xrightarrow{l-1} V$$

Since $M \in \Lambda^m$, there must exist D' and D'' such that $D \equiv (D' \mid D'')$, where D' is a term context, with $T = D'[M]$ and $R = D''[M]$ (the possibilities are: $D' = D$ and $D'' \equiv \mathbf{1}$, if D is a "term context" , or $D' = B$ and $D'' \equiv B^\infty$, if $D = B^\infty$, where B is a term context, or $D \equiv (D' \mid D'')$). Let $C' = D'C_1 \cdots C_{i-1}\langle D''/x \rangle C_{i+1} \cdots C_n$. Then by induction hypothesis we have $C'[N]\Downarrow$, therefore $C[N]\Downarrow$ since $C[N] \rightarrow C'[N]$.

- $C_0 = \lambda x.B$. There are two cases. If $l = 0$ then $C[M]$ is a value, and $C_1[M], \ldots, C_n[M] \in \Sigma$. Since M cannot be a substitution, we clearly have $C_1[N], \ldots, C_n[N] \in \Sigma$, therefore $C[N]$ is a value too. Otherwise $l > 0$ and there exists i $(1 \leq i \leq n)$ such that

$$C[M] \rightarrow B[M]\langle C_i[M]/x \rangle C_1[M] \cdots C_{i-1}[M] C_{i+1}[M] \cdots C_n[M] \xrightarrow{l-1} V$$

Let $C' = B\langle C_i/x \rangle C_1 \cdots C_{i-1} C_{i+1} \cdots C_n$. Then by induction hypothesis we have $C'[N]\Downarrow$, therefore $C[N]\Downarrow$ since $C[N] \rightarrow C'[N]$ ☐

In the rest of the paper, although we sometimes use the symbol $\sqsubseteq$, we shall always regard this preorder as defined by means of applicative tests. An obvious property of the applicative testing preorder is:

$$N \sqsubseteq_{\mathcal{A}} M \Rightarrow \forall P.\; NP \sqsubseteq_{\mathcal{A}} MP \quad \& \quad N\langle P/x \rangle \sqsubseteq_{\mathcal{A}} M\langle P/x \rangle$$

We denote by $\simeq$, or $\simeq_{\mathcal{A}}$, the equivalence associated with the preorder $\sqsubseteq$ ($= \sqsubseteq_{\mathcal{A}}$).

It should be clear that, for a term $MQ_1 \cdots Q_k$, some permutations or garbage collection may occur in the list of arguments and substitutions $Q_1, \ldots, Q_k$ without affecting the potential evaluations. To state this formally, let $\asymp$ be the least relation containing $\equiv \cup =_\alpha$ and satisfying:

$$\begin{array}{rcl} M\langle P/x\rangle \asymp M & & x \notin \mathsf{fv}(M) \\ (MP)\langle R/x\rangle \asymp (M\langle R/x\rangle)P & & x \notin \mathsf{fv}(P) \\ (M\langle P/z\rangle)\langle R/x\rangle \asymp (M\langle R/x\rangle)\langle P/z\rangle & & x \neq z,\ x \notin \mathsf{fv}(P)\ \&\ z \notin \mathsf{fv}(R) \\ M \asymp M' \ \Rightarrow\ (MP) \asymp (M'P) & & \\ M \asymp M' \ \Rightarrow\ (M\langle P/x\rangle) \asymp (M'\langle P/x\rangle) & & \end{array}$$

Then we have:

LEMMA 2.3. $M \asymp N\ \&\ M \to M' \ \Rightarrow\ \exists N'.\ N \to N'\ \&\ M' \asymp N'$

The proof is straightforward $\square$

Note that if $M \asymp N$ and M is a value, then N is a value too. Then an obvious consequence of the previous lemma is:

COROLLARY 2.4. $M \asymp N \ \Rightarrow\ M \simeq_{\mathcal{A}} N$

Another property of applicative testing is that the evaluation process is decreasing w.r.t. this preorder:

LEMMA 2.5. $M \to M' \ \Rightarrow\ M' \sqsubseteq_{\mathcal{A}} M$

The proof is obvious, since $M \to M' \ \Rightarrow\ MQ_1 \cdots Q_k \to M'Q_1 \cdots Q_k$, and $N \to N'\ \&\ N'\Downarrow \ \Rightarrow\ N\Downarrow$. In particular one has $M\langle N/x\rangle \sqsubseteq_{\mathcal{A}} (\lambda x.M)N$, and (combining the two previous facts) if $m > 0$ then $M \sqsubseteq_{\mathcal{A}} x\langle M^m/x\rangle$. It is not difficult to see that the converse inequalities also hold, that is $(\lambda x.M)N \simeq_{\mathcal{A}} M\langle N/x\rangle$ and $x\langle M^m/x\rangle \simeq_{\mathcal{A}} M$ for $m > 0$. One may also note that there is no semantical difference between an empty bag of resources or a bag consisting of an undefined term. That is $M\langle \mathbf{1}/x\rangle \simeq_{\mathcal{A}} M\langle \Omega/x\rangle$ for any M. Similarly, it should be intuitively clear that, the more resources a bag contains, the better it is. More formally:

LEMMA 2.6. *For any M, R and P:*

(i) $M\langle P/x\rangle \sqsubseteq_{\mathcal{A}} M\langle (P \mid R)/x\rangle$

(ii) $MP \sqsubseteq_{\mathcal{A}} M(P \mid R)$

PROOF. To show the first point, we prove that if $T = M\langle P/x\rangle Q_1 \cdots Q_k \xrightarrow{*} V$ for some value V then $S = M\langle (P \mid R)/x\rangle Q_1 \cdots Q_k\Downarrow$, by induction on the length of the reduction $T \xrightarrow{*} V$. If this length is 0 then M is a value, and all the Q_i's are substitution items, therefore S is a value too. Otherwise $T \to T' \xrightarrow{*} V$, and we examine the possible cases for $T \to T'$.

- If $T' = M'\langle P/x\rangle Q_1 \cdots Q_k$ with $M \to M'$ then $S \to S' = M'\langle (P \mid R)/x\rangle Q_1 \cdots Q_k$, therefore by induction hypothesis $S'\Downarrow$, hence $S\Downarrow$.
- If $T' = N\langle Q_i/z\rangle T_1 \cdots T_n\langle P/x\rangle Q_1 \cdots Q_{i-1}Q_{i+1} \cdots Q_k$ with $M = (\lambda z.N)T_1 \cdots T_n$, $Q_i \in \Pi$ and $T_1, \ldots, T_n, Q_1, \ldots, Q_{i-1}$ are subtitution items (not binding any free variable of Q_i) then $S \to S'$ where

$$S' = N\langle Q_i/z\rangle T_1 \cdots T_n\langle (P \mid R)/x\rangle Q_1 \cdots Q_{i-1}Q_{i+1} \cdots Q_k$$

and we use the induction hypothesis.

- The proof is similar if $M = zT_1 \cdots T_n$, $Q_i \equiv \langle (N \mid P')/z \rangle$ and

$$T' = NT_1 \cdots T_n \langle P/x \rangle Q_1 \cdots Q_{i-1} \langle P'/z \rangle Q_{i+1} \cdots Q_k$$

(with some additional requirements that we do not write).

- Finally if $M = xT_1 \cdots T_n$, $P \equiv (N \mid P')$ and

$$T' = NT_1 \cdots T_n \langle P'/x \rangle Q_1 \cdots Q_k$$

(again, omiting some conditions) then $(P \mid R) \equiv (N \mid (P' \mid R))$, and $S \to S'$ where

$$S' = NT_1 \cdots T_n \langle (P' \mid R)/x \rangle Q_1 \cdots Q_k$$

and we use the induction hypothesis.

We proceed in a similar way to show the second point, that is we prove that if

$$T = MPQ_1 \cdots Q_k \xrightarrow{*} V$$

for some value V then $S = M(P|R)Q_1 \cdots Q_k \Downarrow$, by induction on the length of the reduction $T \xrightarrow{*} V$. In the case of $M = (\lambda z.N)T_1 \cdots T_n$ and $T \to T' = N\langle P/z \rangle T_1 \cdots T_n Q_1 \cdots Q_k$ one uses the previous point □

From this lemma it follows that $(M \oplus N)$ is an upper bound of M and N. As a matter of fact, it is not difficult to see that $(M \oplus N)$ is the *join* of M and N:

$$M \sqsubseteq_{\mathcal{A}} T \;\&\; N \sqsubseteq_{\mathcal{A}} T \;\Leftrightarrow\; (M \oplus N) \sqsubseteq_{\mathcal{A}} T$$

To conclude this section, we give some further properties of the applicative testing preorder that will be used later. To state these properties, let us introduce some notations. We use $\rho, \rho', \ldots$ to denote sequences of substitution items, that is $\rho = \langle P_1/x_1 \rangle \cdots \langle P_k/x_k \rangle$. We say that ρ is closed if all the P_i's are closed, and we call the sequence of variables $x_1, \ldots, x_k$ its domain. For $i = 1, \ldots, n$ let $\rho_i = \langle P_1^i/x_1 \rangle \cdots \langle P_k^i/x_k \rangle$ be substitutions with the same domain. Then $(\rho_1 | \cdots | \rho_n)$ denotes the substitution $\langle (P_1^1 \mid \cdots \mid P_1^n)/x_1 \rangle \cdots \langle (P_k^1 \mid \cdots \mid P_k^n)/x_k \rangle$.

We say that a set $\mathcal{R}$ of closed terms of Λ^m is an *applicative simulation* if and only if it satisfies the following conditions:

(i) if $(T, S) \in \mathcal{R}$ and T is a value then S is a value

(ii) if $(T, S) \in \mathcal{R}$ and $T \to T'$ then $S \to S'$ for some S' such that $(T', S') \in \mathcal{R}$.

Clearly if M and N are two terms such that there exists an applicative simulation containing all the pairs of closed terms $(MQ_1 \cdots Q_k, NQ_1 \cdots Q_k)$ then $M \sqsubseteq_{\mathcal{A}} N$. This provides us with a technique for proving $M \sqsubseteq_{\mathcal{A}} N$, that we used in the previous lemma for instance.

LEMMA 2.7. *Let $\rho, \rho_1, \ldots, \rho_n$ be closed substitutions with the same domain, not containing x. Then for any $M, M_1, \ldots, M_n \in \Lambda^m$ the following holds:*

(i) $M\langle (M_1\rho_1 \mid \cdots \mid M_n\rho_n)/x \rangle \rho \sqsubseteq_{\mathcal{A}} (M\langle (M_1 \mid \cdots \mid M_n)/x \rangle)(\rho \mid \rho_1 \mid \cdots \mid \rho_n)$

(ii) $(M\rho)(M_1\rho_1 \mid \cdots \mid M_n\rho_n) \sqsubseteq_{\mathcal{A}} (M(M_1 \mid \cdots \mid M_n))(\rho \mid \rho_1 \mid \cdots \mid \rho_n)$

PROOF. Let $\mathcal{R}$ be the least transitive relation containing set of pairs of closed terms (T, S) such that

- either $T \asymp M\langle (M_1\rho_1 \mid \cdots \mid M_n\rho_n)/x\rangle R_1 \cdots R_m\rho Q_1 \cdots Q_k$ and $S \asymp (M\langle (M_1 \mid \cdots \mid M_n)/x\rangle)R_1 \cdots R_m(\rho \mid \rho_1 \mid \cdots \mid \rho_n)Q_1 \cdots Q_k$, or
- $T \asymp M(M_1\rho_1 \mid \cdots \mid M_n\rho_n)R_1 \cdots R_m\rho Q_1 \cdots Q_k$ and $S \asymp (M(M_1 \mid \cdots \mid M_n))R_1 \cdots R_m(\rho \mid \rho_1 \mid \cdots \mid \rho_n)Q_1 \cdots Q_k$, or
- $T \asymp M\rho R_1 \cdots R_m\rho' Q_1 \cdots Q_k$ and $S \asymp MR_1 \cdots R_m(\rho \mid \rho')Q_1 \cdots Q_k$

where ρ' is a closed substitutions with the same domain as ρ, and $R_1, \ldots, R_m$ are arguments (i.e. elements of Π) or substitution items $\langle R/z\rangle$ such that z is not in the domain of ρ. Then one shows that $\mathcal{R}$ is an applicative simulation. The details are omitted $\square$

2.4 Translation of the Lazy λ-Calculus.

In this section, we show that the λ-calculus with multiplicities is in a sense a strict refinement of the lazy λ-calculus of Abramsky and Ong [2]. Recall that the lazy evaluation of the λ-terms is given by the following two rules:

$$(\lambda x.M)N \to_\ell M[N/x]$$
$$M \to_\ell M' \;\Rightarrow\; MN \to_\ell M'N$$

where $M[N/x]$ is the usual substitution. It is easy to see that:

$$M \to_\ell M' \;\Rightarrow\; M[N/x] \to_\ell M'[N/x]$$

We shall denote by $M \sqsubseteq_\ell N$ the testing preorder in the lazy λ-calculus, where the tests are the ordinary λ-contexts, and the (closed) values are the (closed) normal form for the evaluation process, that is simply the (closed) abstractions $\lambda x.M$. To avoid any confusion, we also denote for a while by $\to_m$ the evaluation in the λ-calculus with multiplicities, and by $M \sqsubseteq_m N$ the testing preorder of our calculus. Now we define a translation $[\![\cdot]\!]^\star$ from λ-terms to terms of Λ^m, as we already suggested, that is:

$$\begin{aligned}
[\![x]\!]^\star &= x \\
[\![\lambda x.M]\!]^\star &= \lambda x.[\![M]\!]^\star \\
[\![MN]\!]^\star &= [\![M]\!]^\star([\![N]\!]^\star)^\infty
\end{aligned}$$

This could be called the "*Girard's translation*". Clearly, the translation actually goes into a subset of $\Lambda^\star$, that we denote Λ^∞, where one only uses infinite multiplicities. Then we can show that this translation is adequate, regardless of the co-domain, while, as a mapping to $\Lambda^\star$, it is not fully abstract (or fully adequate). On the other hand, Λ^∞ obviously provides an exact image of the lazy λ-calculus. In the following proposition, we regard the evaluation of $\lambda^\star$-terms as a binary relation over $\Lambda^\star$. This is true up to structural equivalence, see the Lemma 2.1. Similarly, we regard the evaluation of λ^∞-terms as resulting into λ^∞-terms. Moreover, we denote by $M \sqsubseteq_\star N$ and $M \sqsubseteq_\infty N$ the testing preorders where the tests are confined to the correspoding sub-calculi. Then we can show the following properties of the translation:

PROPOSITION 2.8.

(i) *The translation* $[\![\cdot]\!]^\star$ *is adequate, that is:*

$$[\![M]\!]^\star \sqsubseteq_m [\![N]\!]^\star \ \Rightarrow\ [\![M]\!]^\star \sqsubseteq_\star [\![N]\!]^\star \ \Rightarrow\ [\![M]\!]^\star \sqsubseteq_\infty [\![N]\!]^\star \ \Rightarrow\ M \sqsubseteq_\ell N$$

(ii) *The translation from* Λ *to* $\Lambda^\star$ *is not fully abstract, that is:*

$$\exists M, N \in \Lambda.\ M \sqsubseteq_\ell N \ \&\ [\![M]\!]^\star \not\sqsubseteq_\star [\![N]\!]^\star$$

(iii) *The translation from* Λ *to* Λ^∞ *is fully abstract:*

$$[\![M]\!]^\star \sqsubseteq_\infty [\![N]\!]^\star \ \Leftrightarrow\ M \sqsubseteq_\ell N$$

PROOF. For the first point, we note that since the translation is defined in a compositional way, we just have to show that for a closed λ-term M we have, using obvious notations, $M{\Downarrow_\ell} \Leftrightarrow [\![M]\!]^\star{\Downarrow_m}$. We proceed as for the proof of the similar Theorem 4.6 of Milner [17]. That is, we define the relation $\mathcal{R} \subseteq \Lambda \times \Lambda^m$ as the set of pairs (M,T) such that for some $N_0, N_1, \ldots N_k \in \Lambda$ and distinct variables $x_1, \ldots, x_k$:

(a) $\mathsf{fv}(N_i) \subseteq \{x_{i+1}, \ldots, x_k\}$

(b) $M =_\alpha N_0[N_1/x_1]\cdots[N_k/x_k]$ and $T \asymp [\![N_0]\!]^\star\langle([\![N_1]\!]^\star)^\infty/x_1\rangle\cdots\langle([\![N_k]\!]^\star)^\infty/x_k\rangle$

Then we show that for $(M,T) \in \mathcal{R}$:

(c) if T is a value then M is an abstraction;
  if $T \rightarrow_m T'$ then either $(M,T') \in \mathcal{R}$ or $M \rightarrow_\ell M'$ with $(M',T') \in \mathcal{R}$.

(d) if M is an abstraction then $T \xrightarrow{*}_m V$ for some value V such that $(M,V) \in \mathcal{R}$;
  if $M \rightarrow_\ell M'$ then $T \xrightarrow{*}_m T'$ for some T' such that $(M',T') \in \mathcal{R}$.

(c) The first point is obvious. Now suppose that $T \rightarrow_m T'$. There are two cases: either N_0 is $x_i M_1 \cdots M_n$ and

$$T' \equiv [\![N_i M_1 \cdots M_n]\!]^\star\langle([\![N_1]\!]^\star)^\infty/x_1\rangle\cdots\langle([\![N_k]\!]^\star)^\infty/x_k\rangle$$

or $N_0 =_\alpha (\lambda x_0.M_0)M_1\cdots M_n$ (where x_0 may be assumed to be distinct from $x_1, \ldots, x_k$) and

$$T' \equiv [\![M_0]\!]^\star\langle([\![M_1]\!]^\star)^\infty/x_0\rangle([\![M_2]\!]^\star)^\infty\cdots([\![M_n]\!]^\star)^\infty\langle([\![N_1]\!]^\star)^\infty/x_1\rangle\cdots\langle([\![N_k]\!]^\star)^\infty/x_k\rangle$$

In the first case we have $(M,T') \in \mathcal{R}$. In the second one, we have by definition of $\asymp$:

$$T' \asymp [\![M_0]\!]^\star([\![M_2]\!]^\star)^\infty\cdots([\![M_n]\!]^\star)^\infty\langle([\![M_1]\!]^\star)^\infty/x_0\rangle\langle([\![N_1]\!]^\star)^\infty/x_1\rangle\cdots\langle([\![N_k]\!]^\star)^\infty/x_k\rangle$$

therefore if we let

$$M' = (M_0 M_2 \cdots M_n)[M_1/x_0][N_1/x_1]\cdots[N_k/x_k]$$

we have $M \rightarrow_\ell M'$ (up to α-conversion), and $(M',T') \in \mathcal{R}$.

(d) Assume that M is an abstraction. We show that T has a value V such that $(M,V) \in \mathcal{R}$ by induction on k. For $k = 0$, this is trivial, since then N_0 is an abstraction. Otherwise, for $k > 0$, either N_0 is an abstraction (again, this case is trivial) or N_0 is a variable, say x_i with $1 \leq i \leq k$, and then $M =_\alpha N_i[N_{i+1}/x_{i+1}]\cdots[N_k/x_k]$. We have, using the Lemma 2.3, $T \rightarrow_m T'$ for some T' such that $T' \asymp [\![N_i]\!]^\star\langle([\![N_1]\!]^\star)^\infty/x_1\rangle\cdots\langle([\![N_k]\!]^\star)^\infty/x_k\rangle$. Since $\mathsf{fv}(N_i) \subseteq \{x_{i+1}, \ldots, x_k\}$ we have $T' \asymp [\![N_i]\!]^\star\langle([\![N_{i+1}]\!]^\star)^\infty/x_{i+1}\rangle\cdots\langle([\![N_k]\!]^\star)^\infty/x_k\rangle$ by definition of $\asymp$, therefore $(M,T') \in \mathcal{R}$, and we use the induction hypothesis.

Now suppose that $M \rightarrow_\ell M'$. We have previously noted that T may be written in a unique way as $AQ_1 \cdots Q_h$ where A is either a variable or an abstraction, and $Q_j \in \Pi \cup \Sigma$. Let

$$j(T) = \begin{cases} 0 & \text{if } A \text{ is an abstraction} \\ i & \text{if } A = x_i \end{cases}$$

It is easy to see that if $S \asymp T$ then $j(S) = j(T)$. We show by induction on $j(T)$ that there exists T' such that $T \xrightarrow{*}_m T'$ and $(M', T') \in \mathcal{R}$. If $j(T) = 0$ then $N_0 = (\lambda x_0.N)M_1 \cdots M_n$ where x_0 may be assumed to be distinct from $x_1, \ldots, x_k$, and

$$M' = NM_2 \cdots M_n[M_1/x_0][N_1/x_1] \cdots [N_k/x_k]$$

We let

$$T' = [\![N]\!]^\star \langle ([\![M_1]\!]^\star)^\infty/x_0 \rangle ([\![M_2]\!]^\star)^\infty \cdots ([\![M_n]\!]^\star)^\infty \langle ([\![N_1]\!]^\star)^\infty/x_1 \rangle \cdots \langle ([\![N_k]\!]^\star)^\infty/x_k \rangle$$

Clearly $T \rightarrow_m T'$, and

$$T' \asymp [\![N]\!]^\star ([\![M_2]\!]^\star)^\infty \cdots ([\![M_n]\!]^\star)^\infty \langle ([\![M_1^0]\!]^\star)^\infty/x_0 \rangle \langle ([\![N_1]\!]^\star)^\infty/x_1 \rangle \cdots \langle ([\![N_k]\!]^\star)^\infty/x_k \rangle$$

therefore $(M', T') \in \mathcal{R}$. Otherwise if $j(T) > 0$ we have $N_0 = x_i M_1 \cdots M_n$, and

$$M = (N_i M_1 \cdots M_n)[N_1/x_1] \cdots [N_k/x_k]$$

Then clearly $T \rightarrow_m T'$ for some T' such that

$$T' \asymp [\![N_i M_1 \cdots M_n]\!]^\star \langle ([\![N_1]\!]^\star)^\infty/x_1 \rangle \cdots \langle ([\![N_k]\!]^\star)^\infty/x_k \rangle$$

and we use the induction hypothesis, since $(M, T') \in \mathcal{R}$, and $\mathsf{fv}(N_i) \subseteq \{x_{i+1}, \ldots, x_k\}$.

To show the second point of the proposition, we have to exhibit a pair of λ-terms that are distinguished using contexts of the λ-calculus with multiplicities, but not by λ-calculus contexts. Let $\Delta' = \lambda x(x(\lambda y.xy))$. Then $\Delta = \lambda x(xx)$ and Δ' have the same interpretation in the canonical model of the lazy λ-calculus, solution of the domain equation $D = (D \rightarrow D)_\perp$ (see [2,6] for this interpretation). Therefore they are operationally indistinguishable, that is $\Delta \simeq_\ell \Delta'$, since this interpretation of the lazy λ-calculus is adequate. However, as we have seen, $([\![\Delta]\!]^\star\, \mathbf{I})\Uparrow_m$, while

$$([\![\Delta']\!]^\star\, \mathbf{I}) \xrightarrow{*} \lambda y.(xy^\infty)\langle \lambda y.(xy^\infty)^\infty/z \rangle \langle \mathbf{1}/x \rangle$$

that is $([\![\Delta']\!]^\star\, \mathbf{I})\Downarrow_m$.

The proof of the last point is trivial $\square$

One may observe that we have proved a little more than the "non-full abstraction" of the translation from Λ to Λ^m: since the two terms Δ and Δ' have the same interpretation in $D = (D \rightarrow D)_\perp$, they are not distinguished by adding a convergence testing facility, or any other construct for which this domain provides an adequate interpretation, like parallel composition or non-deterministic choice (see [2,5,6]). Therefore, if we were to define a translation from a λ-calculus extended with these constructs into Λ^m, this translation would still be non-fully abstract. Obviously we do not regard this fact as a defect. Indeed, the λ-calculus with multiplicities was conceived exactly for the purpose of distinguishing terms like Δ and Δ', by means of a refined management of the arguments of a function, like it arises from Milner's encoding of the λ-calculus into the π-calculus.

3. Semantics.

3.1 The Functionality Theory.

In this section we introduce a refinement of the "intersection type discipline" of Coppo *et al.* [7,8,9], which will serve to give a semantics to our calculus. We recall that the grammar for functional characters is, as given in the introduction:

$$\phi ::= \omega \mid (\pi \to \phi)$$
$$\pi ::= \phi \mid (\pi \times \pi)$$

The set of formulae of the first kind is denoted Φ. They will be used as functional characters for terms of Λ^m. On the other hand, the formulae of the second kind are used to type the bags of resources. We denote by Π the set of these formulae. We shall use ϕ, $\psi \ldots$ and π, $\zeta \ldots$ to range over Φ and Π respectively, and we use τ, $\sigma \ldots$ to denote formulae of any kind. We abbreviate $(\pi_1 \times (\cdots \times (\pi_n \times \pi_{n+1}) \cdots))$ as $\pi_1 \times \cdots \times \pi_n \times \pi_{n+1}$, and, as usual, $\pi_1 \to \cdots \to \pi_n \to \phi$ is an abbreviation for $(\pi_1 \to (\cdots \to (\pi_n \to \phi) \cdots))$.

REMARKS. Our results would still hold if we were to add an *additive conjunction* $\phi \wedge \psi$ for formulae of Φ. Similarly, we could use Girard's exponential "of course" to build formulae $!\phi$ of Π from formulae of Φ, that can be used to type terms with an infinite multiplicity. However, since we can content ourselves with "compact" functional characters, we do not need this construct (see the Approximation Theorem of Girard [11]). Indeed, one may remark that the construct M^∞ is not mentioned in the evaluation rules E1-E5, except indirectly in E5, by means of $M^\infty \equiv (M \mid M^\infty)$.

The typing system, or more appropriately the *functionality system* is an intuitionistic natural deduction system, presented in sequent form. The sequents are either $x_1 : \pi_1, \ldots, x_k : \pi_k \vdash M : \phi$, for typing terms of Λ^m, or $x_1 : \pi_1, \ldots, x_k : \pi_k \vdash P : \pi$, for typing bags of ressources. As usual, we use Γ, $\Delta \ldots$ to denote hypotheses, that is sequences $x_1 : \pi_1, \ldots, x_k : \pi_k$, where a variable may occur more than once. When the hypothesis is empty, we write $\vdash T : \tau$. Note that although the variables are terms of Λ^m, the hypotheses we formulate about them are assertions concerning bags of resources. Then $x : \pi$ may be read as "x is one of the resources of a bag satisfying π".

Since our typing system – as opposed to the various "linear term calculi" one finds in the literature ([3,4,15,16,21]) – does not record the various manipulations of the hypothesis – like weakenings, contractions, derelictions, … – as terms constructions, we shall factorize these manipulations into just one rule. For this purpose, we write $\Gamma \gg \Delta$ whenever the hypothesis Δ results from Γ by application of a sequence of exchange, weakening, or product. That is, $\gg$ is the least preorder satisfying:

$$\Gamma, x : \pi, y : \zeta, \Delta \gg \Gamma, y : \zeta, x : \pi, \Delta \qquad \textit{exchange}$$
$$\Gamma \gg x : \pi, \Gamma \qquad \textit{weakening}$$
$$x : \pi, x : \zeta, \Gamma \gg x : \pi \times \zeta, \Gamma \qquad \textit{product}$$

The first group of typing rules concerns the constructions of the calculus:

$$\text{L1:} \quad x : \phi \vdash x : \phi \qquad\qquad \text{L2:} \quad \frac{x : \pi, \Gamma \vdash M : \phi}{\Gamma \vdash \lambda x.M : \pi \to \phi} \ (x \text{ not in } \Gamma)$$

$$\text{L3:} \quad \frac{\Gamma \vdash M : \pi \to \phi \ , \ \Delta \vdash P : \pi}{\Gamma, \Delta \vdash (MP) : \phi} \qquad\qquad \text{L4:} \quad \frac{\Gamma \vdash P : \pi \ , \ x : \pi, \Delta \vdash M : \phi}{\Gamma, \Delta \vdash M\langle P/x \rangle : \phi} \ (x \text{ not in } \Delta)$$

$$\text{L5:}\quad \frac{\Gamma \vdash P:\pi\ ,\ \Delta \vdash R:\zeta}{\Gamma, \Delta \vdash (P \mid R):\pi \times \zeta} \qquad\qquad \text{L6:}\quad \frac{\Gamma \vdash (M \mid M^{\infty}):\pi}{\Gamma \vdash M^{\infty}:\pi}$$

The apparent circularity in the last rule, L6 for M^{∞}, may in fact be broken by using the rule L7 below. The remaining rules of our system are independent from the structure of the terms. For technical convenience, we shall introduce a rule partly reflecting the structural equivalence $T \equiv S$ in the typing system. To this end, let us define the congruence $\sim$ over formulae as the least one satisfying:

$$\omega \times \pi \sim \pi$$

$$\pi_0 \times \pi_1 \sim \pi_1 \times \pi_0$$

$$\pi_0 \times (\pi_1 \times \pi_2) \sim (\pi_0 \times \pi_1) \times \pi_2$$

$$\pi \sim \zeta \ \&\ \phi \sim \psi \ \Rightarrow\ \pi \to \phi \sim \zeta \to \psi$$

Then our last rules are:

$$\text{L7:}\quad \Gamma \vdash P:\omega \qquad\qquad \text{L8:}\quad \frac{\Gamma \vdash T:\tau}{\Delta \vdash T:\tau}\ \Gamma \gg \Delta \qquad\qquad \text{L9:}\quad \frac{\Gamma \vdash T:\tau}{\Gamma \vdash T:\sigma}\ \tau \sim \sigma$$

In the rule L9 we implicitly assume that σ is in Φ if $T \in \Lambda^m$. For instance, although $\omega \times \omega \sim \omega$, we do not allow using L9 to infer $\vdash M:\omega \times \omega$.

Now let us make some comments on our functionality system. The first three rules are quite standard. The rule L4 associate explicit substitutions with the logical "cut rule", as in [3] for instance. The way we accumulate the hypotheses regarding the subterms in the rules L3-L5 is typical of a "multiplicative" management, according to Girard's terminology (in particular, L5 is the rule for introducing the multiplicative conjunction). Since we allow the usual weakening rule, this discipline may be derived from the usual "additive" one. However, since we disallow contraction, it is strictly more restrictive. If for instance we were to use, instead of L3:

$$\frac{\Gamma \vdash M:\pi \to \phi\ ,\ \Gamma \vdash P:\pi}{\Gamma \vdash (MP):\phi}$$

then we could infer $x:\phi, y:\phi \to (\phi \to \psi) \vdash (yx)x:\psi$, which is not possible in our system. The additive management of the hypotheses involves implicit contractions that we wish to avoid. Apart from this, the main difference with Coppo and Dezani's typing system is the rule L5, introducing a term constructor, unlike the usual rule, which is:

$$\frac{\Gamma \vdash M:\phi\ ,\ \Gamma \vdash M:\psi}{\Gamma \vdash M:\phi \wedge \psi}$$

This rule may be regarded as a "contraction rule" - contracting the terms, not the formulae. We have indicated that the results we will prove still hold if one uses also this "additive" conjunction $\phi \wedge \psi$. This refers to the usual rules for this connective, that is, apart from the previous one:

$$\frac{\Gamma \vdash M:\phi \wedge \psi}{\Gamma \vdash M:\phi} \qquad \frac{\Gamma \vdash M:\phi \wedge \psi}{\Gamma \vdash M:\psi}$$

Now let us see some examples. As with the intersection type discipline, one may assign a meaningful functional character to the duplicator $\Delta = \lambda x(xx^\infty)$, as follows:

$$
\text{L2}\ \dfrac{\text{L8}\ \dfrac{\text{L3}\ \dfrac{\text{L1}\ \dfrac{}{x:\phi \to \psi \vdash x:\phi \to \psi} \qquad \text{L9}\ \dfrac{\text{L6}\ \dfrac{\text{L5}\ \dfrac{\text{L1}\ \dfrac{}{x:\phi \vdash x:\phi} \qquad \text{L7}\ \dfrac{}{\vdash x^\infty:\omega}}{x:\phi \vdash (x \mid x^\infty):\phi \times \omega}}{x:\phi \vdash x^\infty:\phi \times \omega}}{x:\phi \vdash x^\infty:\phi}}{x:\phi \to \psi, x:\phi \vdash (xx^\infty):\psi}}{x:(\phi \to \psi) \times \phi \vdash (xx^\infty):\psi}}{\vdash \lambda x(xx^\infty):((\phi \to \psi) \times \phi) \to \psi}
$$

Clearly one has $\vdash \lambda x.x:\phi \to \phi$, therefore if we let $\phi = \psi = \tau \to \tau$ we have:

$$
\text{L3}\ \dfrac{\dfrac{\vdots}{\vdash \lambda x(xx^\infty):((\phi \to \psi) \times \phi) \to \psi} \qquad \text{L5}\ \dfrac{\dfrac{\vdots}{\vdash \lambda x.x:\phi \to \psi} \qquad \dfrac{\vdots}{\vdash \lambda x.x:\phi}}{\vdash (\lambda x.x)^2:(\phi \to \psi) \times \phi}}{\vdash (\lambda x(xx^\infty))(\lambda x.x)^2:\tau \to \tau}
$$

Regarding the combinator $\mathsf{S} = \lambda xyz(xz^\infty)(yz^\infty)^\infty$, one can infer the following typing:

$$\vdash \mathsf{S}:(\pi \to \psi \to \phi) \to (\zeta \to \psi) \to (\pi \times \zeta) \to \phi$$

The following example shows that $M \oplus N$ has the types of M (and similarly the types of N):

$$
\text{L4}\ \dfrac{\text{L9}\ \dfrac{\text{L5}\ \dfrac{\dfrac{\vdots}{\Gamma \vdash M:\phi} \qquad \text{L7}\ \dfrac{}{\vdash N:\omega}}{\Gamma \vdash (M \mid N):\phi \times \omega}}{\Gamma \vdash (M \mid N):\phi} \qquad \text{L1}\ \dfrac{}{x:\phi \vdash x:\phi}}{\Gamma \vdash x\langle (M \mid N)/x \rangle:\phi}
$$

To conclude this section we may now define the semantics of our calculus, using the functionality system:

DEFINITION. *$\mathcal{F}[\![M]\!]$ is the set of pairs (Γ, ϕ) such that $\Gamma \vdash M:\phi$.*

We shall write $N \sqsubseteq_{\mathcal{F}} M$ whenever $\mathcal{F}[\![N]\!] \subseteq \mathcal{F}[\![M]\!]$. Our aim is now to show the following result:

ADEQUACY THEOREM.

$$\mathcal{F}[\![N]\!] \subseteq \mathcal{F}[\![M]\!] \Rightarrow N \sqsubseteq M$$

The rest of the paper is devoted to establish this result.

3.2 The Adequacy Theorem: Reduction is Decreasing.

It should be clear that the preorder $N \sqsubseteq_{\mathcal{F}} M$ is compatible with the constructions of the calculus, that is

$$N \sqsubseteq_{\mathcal{F}} M \;\Rightarrow\; \forall C.\; C[N] \sqsubseteq_{\mathcal{F}} C[M]$$

Therefore to prove the theorem we only have to prove the following, for closed terms:

$$N \sqsubseteq_{\mathcal{F}} M \;\&\; N\Downarrow \;\Rightarrow\; M\Downarrow$$

This will be established by showing the property of *computational adequacy*, stating that a closed term M is convergent if and only if it has a non-trivial type, namely:

$$M\Downarrow \;\Leftrightarrow \vdash M : \pi \to \phi$$

for some π and ϕ. To prove the "$\Rightarrow$" direction, we show that any value has a non-trivial functional character, and that the typing is preserved by "expansion", that is:

(i) $\vdash V : \pi \to \omega$

(ii) $M \to N \;\&\; \Gamma \vdash N : \phi \;\Rightarrow\; \Gamma \vdash M : \phi$

LEMMA 3.1. $\vdash V : \pi \to \omega$

PROOF. The proof is given by the following deductions:

$$\dfrac{\dfrac{}{x : \pi \vdash M : \omega}\,\text{L7}}{\vdash \lambda x.M : \pi \to \omega}\,\text{L2} \qquad \text{and} \qquad \dfrac{\dfrac{}{\vdash P : \omega}\,\text{L7} \qquad \dfrac{\dfrac{\vdots}{\vdash V : \pi \to \omega}}{x : \omega \vdash V : \pi \to \omega}\,\text{L8}}{\vdash V\langle P/x\rangle : \pi \to \omega}\,\text{L4}$$

We shall prove a slightly more general result than (ii) above, by considering a notion of *reduction* that contains the evaluation relation, and may be used to define other evaluation strategies. Roughly, this relation allows evaluating in any context, . In particular, reductions may occur within P in the terms MP and $M\langle P/x\rangle$, and reductions may take place within an abstraction. Moreover, the use of a substitution item $\langle P/x\rangle$ is not limited to the case where the variable appears in the head position. In particular, $(\lambda x.M)\langle P/z\rangle$ may be reduced as $\lambda x.(M\langle P/z\rangle)$, provided this does not induce any capture of free variables. The reduction relation $T \rhd T'$, defined using the auxiliary relation $M\langle N/x\rangle \Vdash\!\rhd M'$, is the least relation satisfying the evaluation rules E1-E5 (replacing $\to$ by $\rhd$ and $\mapsto$ by $\Vdash\!\rhd$), and:

$$\text{R1:}\quad x \neq z \;\&\; x \notin \mathsf{fv}(P) \;\Rightarrow\; (\lambda x.M)\langle P/z\rangle \rhd \lambda x.(M\langle P/z\rangle) \qquad \text{R2:}\quad \dfrac{M \rhd M'}{\lambda x.M \rhd \lambda x.M'}$$

$$\text{R3:}\quad \dfrac{P \rhd P'}{MP \rhd MP'} \qquad \text{R4:}\quad \dfrac{P \rhd P'}{M\langle P/x\rangle \rhd M\langle P'/x\rangle}$$

$$\text{R5:}\quad \dfrac{M \rhd M'}{(M \mid P) \rhd (M' \mid P)} \qquad \text{R6}\quad \dfrac{P \rhd P'}{Q \rhd P'}\; Q \equiv P$$

where $\Rrightarrow$ is the least relation satisfying S1-S3 (replacing $\mapsto$ by $\Rrightarrow$) and:

$$\text{S4:}\quad \frac{M\langle {}^{N}/_{x}\rangle \Rrightarrow M'}{(R(M \mid P))\langle {}^{N}/_{x}\rangle \Rrightarrow R(M' \mid P)} \qquad\qquad \text{S5:}\quad \frac{M\langle {}^{N}/_{x}\rangle \Rrightarrow M'}{(R\langle (M \mid P)/z\rangle)\langle {}^{N}/_{x}\rangle \Rrightarrow R\langle (M' \mid P)/z\rangle}$$

It should be obvious that $M\langle {}^{N}/_{x}\rangle \mapsto M' \;\Rightarrow\; M\langle {}^{N}/_{x}\rangle \Rrightarrow M'$, and $M \to M' \;\Rightarrow\; M \rhd M'$. Then we shall show the property known as "subject expansion", that is:

$$T \rhd T' \;\&\; \Gamma \vdash T' : \tau \;\Rightarrow\; \Gamma \vdash T : \tau$$

For the terms of Λ^m, this property may also be called "*reduction is decreasing*", since it may be read:

$$M \rhd M' \;\Rightarrow\; \mathcal{F}[\![M']\!] \subseteq \mathcal{F}[\![M]\!]$$

We should first check that:

$$T =_\alpha T' \;\&\; \Gamma \vdash T' : \tau \;\Rightarrow\; \Gamma \vdash T : \tau$$

This will be omitted. Then a first step towards the "subject expansion" property is:

LEMMA 3.2. $P \equiv P' \;\&\; \Gamma \vdash P' : \pi \;\Rightarrow\; \Gamma \vdash P : \pi$

PROOF. One proceeds by induction on the definition of $P \equiv P'$, and then on the inference of $\Gamma \vdash P' : \pi$, and by case on the last rule used in this inference. The details are omitted ❑

Next we need some technical lemmas concerning the functionality system. Clearly the only way to introduce in the hypothesis of a sequent $\Gamma \vdash T : \tau$ some variables which are not free in the term T is by means of L7 or L8. Therefore we have:

LEMMA 3.3. *If* $\Gamma, x : \pi, \Delta \vdash T : \tau$, *where* $x \notin \mathsf{fv}(T)$, *then* $\Gamma, \Delta \vdash T : \tau$.

The proof is straightforward ❑

The next lemma states that the typings of compound terms of the calculus are essentially given by using the rules L2-L6.

LEMMA 3.4. *Let* $\phi \neq \omega$.

(i) *if* $\Gamma \vdash \lambda x.M : \phi$ *then for some* π, ψ *and* Γ' *not containing* x *we have* $x : \pi, \Gamma' \vdash M : \psi$ *with* $\Gamma' \gg \Gamma$ *and* $\phi \sim \pi \to \psi$

(ii) *if* $\Gamma \vdash MP : \phi$ *then* $\Gamma' \vdash M : \pi \to \phi$ *and* $\Gamma'' \vdash P : \pi$ *for some* π *and* Γ', Γ'' *such that* $\Gamma', \Gamma'' \gg \Gamma$

(iii) *if* $\Gamma \vdash M\langle {}^{P}/_{x}\rangle : \phi$ *then* $\Gamma' \vdash P : \pi$ *and* $x : \pi, \Gamma'' \vdash M : \phi$ *for some* π *and* Γ', Γ'' *such that* $\Gamma', \Gamma'' \gg \Gamma$ *and* x *is not in* Γ''

(iv) *if* $\Gamma \vdash (P \mid R) : \pi$ *then* $\Gamma' \vdash P : \pi_0$ *and* $\Gamma'' \vdash R : \pi_1$ *for some* Γ', Γ'', π_0 *and* π_1 *such that* $\Gamma', \Gamma'' \gg \Gamma$ *and* $\pi \sim \pi_0 \times \pi_1$.

PROOF. By induction on the inference of the sequents, straightforward ❑

Note that the point (iii) of the lemma is similar to the "paste property" of the intersection type discipline.

COROLLARY 3.5. *If* $x \notin \mathsf{fv}(P)$ *then* $\Gamma \vdash (M\langle {}^{R}/_{x}\rangle)P : \phi \;\Leftrightarrow\; \Gamma \vdash (MP)\langle {}^{R}/_{x}\rangle : \phi$.

PROOF. This is trivial for $\phi = \omega$, Then, assuming $\phi \neq \omega$, we only prove the "$\Leftarrow$" direction – the converse implication is similar, and even simpler. By the previous lemma, for some π, Γ' and Γ'' we have:

$$\Gamma' \vdash R : \pi \qquad \text{and} \qquad x : \pi, \Gamma'' \vdash MP : \phi$$

with $\Gamma', \Gamma'' \gg \Gamma$, and x is not in Γ''. By the Lemma 3.4 again, there exist ζ, Δ and Δ' such that:

$$\Delta \vdash M : \zeta \to \phi \qquad \text{and} \qquad \Delta' \vdash P : \zeta$$

with $\Delta, \Delta' \gg x : \pi, \Gamma''$. Since $x \notin \mathsf{fv}(P)$ by the Lemma 3.3 there exists Δ'' not containing x such that $\Delta'' \vdash P : \zeta$ and $\Delta'' \gg \Delta'$, therefore $\Delta, \Delta'' \gg x : \pi, \Gamma''$. It is easy to check that there exists Ξ not containing x such that $\Delta \gg x : \pi, \Xi$ and $\Xi, \Delta'' \gg \Gamma''$. Therefore we get:

$$\text{L4}\ \dfrac{\dfrac{\vdots}{\Gamma' \vdash R : \zeta} \qquad \text{L8}\ \dfrac{\dfrac{\vdots}{\Delta \vdash M : \zeta \to \phi}}{x : \pi, \Xi \vdash M : \zeta \to \phi}}{\Gamma', \Xi \vdash M\langle R/x \rangle : \zeta \to \phi}$$

Finally by L3 and L8 we have $\Gamma \vdash (M\langle R/x \rangle)P : \phi$ $\square$

To prove "subject expansion" we need a similar property regarding the auxiliary "fetch" relation, namely:

LEMMA 3.6. *If $M\langle N/x \rangle \rhd M'$ and $\Gamma \vdash M' : \phi$ then there exist ψ, Γ' and Γ'' such that $\Gamma' \vdash N : \psi$ and $x : \psi, \Gamma'' \vdash M : \phi$ with $\Gamma', \Gamma'' \gg \Gamma$.*

PROOF. By induction on the proof of $M[N/x] \rhd M'$, and then on the inference of $\Gamma \vdash M' : \phi$. We note first that the lemma is obvious if $\phi = \omega$: in this case we let $\psi = \omega$. Then $\vdash N : \omega$ and $x : \omega, \Gamma \vdash M : \omega$ by L7. Therefore we shall assume that $\phi \neq \omega$ in the following.

(S1) If $M = x$ and $M' = N$, then we let Γ'' be the empty hypothesis, $\Gamma' = \Gamma$ and $\psi = \phi$. We have $x : \varphi \vdash x : \varphi$ by L1.

(S2) If $M = RP$ and $M' = R'P$ with $R\langle N/x \rangle \rhd R'$, then by the Lemma 3.4 we have $\Gamma_0 \vdash P : \pi$ and $\Gamma_1 \vdash R' : \pi \to \phi$ with $\Gamma_0, \Gamma_1 \gg \Gamma$. By induction hypothesis, there exist ψ, Γ' and Δ such that $\Gamma' \vdash N : \psi$ and $x : \psi, \Delta \vdash R : \pi \to \phi$ with $\Gamma', \Delta \gg \Gamma_1$. Then by L3 we have:

$$x : \psi, \Delta, \Gamma_0 \vdash M : \phi$$

We let $\Gamma'' = \Delta, \Gamma_0$. Then $\Gamma', \Gamma'' = \Gamma', \Delta, \Gamma_0 \gg \Gamma_0, \Gamma_1 \gg \Gamma$.

(S3) If $M = R\langle P/z \rangle$ and $M' = R'\langle P/z \rangle$ with $R\langle N/x \rangle \rhd R'$ and $z \notin \mathsf{fv}(N) \cup \{x\}$, then by the Lemma 3.4 we have $\Gamma_0 \vdash P : \pi$ and $z : \pi, \Gamma_1 \vdash R' : \phi$ for some Γ_0, Γ_1 such that $\Gamma_0, \Gamma_1 \gg \Gamma$, with z not in Γ_1. By induction hypothesis, there exist ψ, Δ and Δ' such that $\Delta \vdash N : \psi$ and $x : \psi, \Delta' \vdash R : \phi$, with $\Delta, \Delta' \gg z : \pi, \Gamma_1$. Since $z \notin \mathsf{fv}(N)$ by the Lemma 3.3 there exists Γ' not containing z such that $\Gamma' \vdash N : \psi$ and $\Gamma' \gg \Delta$, therefore $\Gamma', \Delta' \gg z : \pi, \Gamma_1$. It is easy to check that there exists Ξ not containing z such that $\Delta' \gg z : \pi, \Xi$ and $\Gamma', \Xi \gg \Gamma_1$. We let $\Gamma'' = \Gamma_0, \Xi$. Then clearly $\Gamma', \Gamma'' \gg \Gamma$, and $x : \psi, \Gamma'' \vdash M : \phi$ by L8 and L4.

(S4) If $M = (R(L \mid P))$ and $M' = R(L' \mid P)$ with $L\langle N/x \rangle \rhd L'$ then by the Lemma 3.4 there exist π, Γ_0 and Γ_1 such that:

$$\Gamma_0 \vdash (L' \mid P) : \pi \qquad \text{and} \qquad \Gamma_1 \vdash R : \pi \to \phi$$

with $\Gamma_0, \Gamma_1 \gg \Gamma$. By the Lemma 3.4 again, there exist Δ_0, Δ_1, π_0 and π_1 such that:

$$\Delta_0 \vdash L' : \pi_0 \qquad \text{and} \qquad \Delta_1 \vdash P : \pi_1$$

with $\Delta_0, \Delta_1 \gg \Gamma_0$ and $\pi \sim \pi_0 \times \pi_1$. Then by induction hypothesis, there exist Γ', Δ and ψ such that $\Gamma' \vdash N : \psi$ and $x : \psi, \Delta \vdash L : \pi_0$, with $\Gamma', \Delta \gg \Delta_0$. We let $\Gamma'' = \Delta, \Delta_1, \Gamma_1$. Clearly $\Gamma', \Gamma'' \gg \Gamma$, and we have:

$$\dfrac{\dfrac{\vdots}{\Gamma_1 \vdash R : \pi \to \phi} \quad \dfrac{\dfrac{\dfrac{\vdots}{x : \psi, \Delta \vdash L : \pi_0} \quad \dfrac{\vdots}{\Delta_1 \vdash P : \pi_1}}{x : \psi, \Delta, \Delta_1 \vdash (L \mid P) : \pi_0 \times \pi_1}\,\text{L5}}{x : \psi, \Delta, \Delta_1 \vdash (L \mid P) : \pi}\,\text{L9}}{x : \psi, \Gamma'' \vdash M : \phi}\,\text{L3}$$

(S5) We omit this case, which is similar to the previous one $\square$

Now we are ready to establish the "subject expansion" property:

PROPOSITION (SUBJECT EXPANSION) 3.7. $T \rhd T' \;\&\; \Gamma \vdash T' : \tau \;\Rightarrow\; \Gamma \vdash T : \tau$

PROOF. By induction on the proof of $T \rhd T'$, and then on the proof of $\Gamma \vdash T' : \tau$. We immediately note that one can "factorize" the use of the rules L7-L9: if $\Gamma \vdash T' : \tau$ is proved by an application of one of these rules, one obviously applies the induction hypothesis, and then the same rule. Then we shall not take these rules into consideration in the following. We proceed by case on the last rule used to infer $T \rhd T'$. We shall only examine in details a few cases.

(E1,E2,R2,R3,R4,R5) These cases are obvious: the last rule used to infer the sequent $\Gamma \vdash T' : \tau$ may be assumed to be L3, L4, L2, L3, L4, L5 respectively. Then one uses the induction hypothesis, and the same rule to infer $\Gamma \vdash T : \tau$.

(E3) If $T = (\lambda x.M)P$ and $T' = M\langle P/x\rangle$ then, assuming that the last rule used to infer $\Gamma \vdash T' : \tau$ is L4, we have $\Gamma' \vdash P : \pi$ and $x : \pi, \Gamma'' \vdash M : \phi$ for some π, where $\Gamma = \Gamma', \Gamma''$ and x is not in Γ''. Then using L2 and L3 we get $\Gamma \vdash T : \phi$. That is, we have the usual picture:

$$\dfrac{\dfrac{\vdots}{\Gamma \vdash P : \pi} \quad \dfrac{\vdots}{x : \pi, \Delta \vdash M : \phi}}{\Gamma, \Delta \vdash M\langle N/x\rangle : \phi} \quad \text{expands to} \quad \dfrac{\dfrac{\vdots}{\Gamma \vdash P : \pi} \quad \dfrac{\dfrac{\vdots}{x : \pi, \Delta \vdash M : \phi}}{\Delta \vdash \lambda x.M : \pi \to \phi}}{\Gamma, \Delta \vdash (MP) : \phi}$$

(E4) In this case one may assume that the last rule used to infer $\Gamma \vdash T' : \tau$ is L4. Then one uses the induction hypothesis and the Corollary 3.5.

(E5) We have $T = M\langle R/x\rangle$ and $T' = M'\langle P/x\rangle$ with $R \equiv (N \mid P)$ and $M\langle N/x\rangle \Rrightarrow M'$. Again, one may assume that the last rule used to infer $\Gamma \vdash T' : \tau$ in this case is L4, that is:

$$\Gamma' \vdash P : \pi \qquad \text{and} \qquad x : \pi, \Gamma'' \vdash M' : \tau$$

with $\Gamma = \Gamma', \Gamma''$, and x is not in Γ''. Then by the Lemma 3.6 we have:

$$\Delta \vdash N : \psi \qquad \text{and} \qquad x : \psi, \Delta' \vdash M : \tau$$

for some ψ, Δ and Δ' such that $\Delta, \Delta' \gg x : \pi, \Gamma''$. Since $x \notin \mathsf{fv}(N)$ by the Lemma 3.3 there exists Ξ not containing x such that $\Xi \vdash N : \psi$ and $\Xi \gg \Delta$. Moreover it is easy to check that there exists Δ'' such that $\Delta' \gg x : \pi, \Delta''$ and $\Xi, \Delta'' \gg \Gamma''$. By L5 we have $\Xi, \Gamma' \vdash (N \mid P) : \psi \times \pi$, hence also $\Xi, \Gamma' \vdash R : \psi \times \pi$ by the Lemma 3.2. Therefore $\Gamma \vdash T : \tau$ can be inferred in the following way:

$$\text{L8}\;\dfrac{\text{L4}\;\dfrac{\dfrac{\vdots}{\Xi, \Gamma' \vdash R : \psi \times \pi} \qquad \text{L8}\;\dfrac{\text{L8}\;\dfrac{\dfrac{\vdots}{x : \psi, \Delta' \vdash M : \tau}}{x : \psi, x : \pi, \Delta'' \vdash M : \tau}}{x : \psi \times \pi, \Delta'' \vdash M : \tau}}{\Xi, \Gamma', \Delta'' \vdash M\langle R/x \rangle : \tau}}{\Gamma \vdash M\langle R/x \rangle : \tau}$$

(R1) If $T' = \lambda x.(M\langle P/z \rangle)$ and $T = (\lambda x.M)\langle P/z \rangle$ with $z \neq x$ and $x \notin \mathsf{fv}(P)$ then, assuming that the sequent $\Gamma \vdash T' : \tau$ is proved using L2 we have $\tau = \pi \to \phi$ with:

$$x : \pi, \Gamma \vdash M\langle P/z \rangle : \phi$$

where Γ does not contain x. By the Lemma 3.4 we get:

$$\Gamma' \vdash P : \zeta \qquad \text{and} \qquad z : \zeta, \Gamma'' \vdash M : \phi$$

where $\Gamma', \Gamma'' \gg \Gamma$, and z is not in Γ''. Since $x \notin \mathsf{fv}(P)$ by the Lemma 3.3 there exists Ξ not containing x such that $\Xi \vdash P : \zeta$ and $\Xi \gg \Gamma'$. Moreover it is easy to check that there exists Δ not containing z such that $\Gamma'' \gg x : \pi, \Delta$ and $\Xi, \Delta \gg \Gamma$ (hence Δ does not contain x). Therefore one can build a proof of $\Gamma \vdash T : \tau$ as follows:

$$\text{L8}\;\dfrac{\text{L4}\;\dfrac{\dfrac{\vdots}{\Xi \vdash P : \zeta} \qquad \text{L2}\;\dfrac{\text{L8}\;\dfrac{\text{L8}\;\dfrac{\dfrac{\vdots}{z : \zeta, \Gamma'' \vdash M : \phi}}{z : \zeta, x : \pi, \Delta \vdash M : \phi}}{x : \pi, z : \zeta, \Delta \vdash M : \phi}}{z : \zeta, \Delta \vdash \lambda x.M : \pi \to \phi}}{\Xi, \Delta \vdash (\lambda x.M)\langle P/z \rangle : \pi \to \phi}}{\Gamma \vdash T : \tau}$$

(R6) This case is obvious, using the induction hypothesis and the Lemma 3.2 ❑

An obvious corollary of the Proposition 3.7 and the Lemma 3.1 is a first half of the computational adequacy property, that is:

COROLLARY 3.8. $M\Downarrow \;\Rightarrow\; \vdash M : \omega \to \omega$

3.3 The Adequacy Theorem: Realizability and Soundness.

To prove the converse of the previous implication, we use Tait's technique, that is we introduce a predicate of *realizability*, which is a set $\mathcal{R}$ of pairs (T,τ) where T is a closed term, and τ is a formula of appropriate kind. We shall write $(T,\tau)\in\mathcal{R}$ as $\models T:\tau$ (in Tait's notation this would be written $\mathcal{R}_\tau(T)$). This may be read "T realizes, or satisfies τ". To define the realizability predicate for the formulae of Π, we use the following observation: any formulae $\pi\in\Pi$ may be written, up to $\sim$, as a product $\phi_1\times\cdots\times\phi_n$ of formulae of Φ. More precisely, let us define the relation (which is in fact a mapping) $\triangleright$ from formulae of Π to sequences of formulae of Φ, as follows:

(i) $\omega\triangleright\varepsilon$ (the empty sequence) and $\pi\to\phi\triangleright\pi\to\phi$

(ii) $\pi\triangleright\phi_1,\ldots,\phi_n$ & $\zeta\triangleright\psi_1,\ldots,\psi_m \;\Rightarrow\; (\pi\times\zeta)\triangleright\phi_1,\ldots,\phi_n,\psi_1,\ldots,\psi_m$

The following should be clear :

REMARK 3.9. *$\pi\sim\zeta$ if and only if $\pi\triangleright\phi_1,\ldots,\phi_n$ and $\zeta\triangleright\psi_1,\ldots,\psi_m$ implies $m=n$ and there exists a permutation i of $1,\ldots,n$ such that $\phi_{i_j}\sim\psi_j$.*

Then we define the realizability predicate $\models T:\tau$, by induction on the formulae, as follows:

$$\begin{aligned}
\models M:\omega &\Leftrightarrow_{\rm def} \text{true}\\
\models M:\pi\to\phi &\Leftrightarrow_{\rm def} M\Downarrow \;\;\&\;\; \forall P\,(\models P:\pi \Rightarrow\, \models (MP):\phi)\\
\models P:\pi &\Leftrightarrow_{\rm def} \pi\triangleright\phi_1,\ldots,\phi_n \,\&\, n>0 \Rightarrow\\
&\qquad\qquad \exists R\,\forall i\,\exists M_i.\; \models M_i:\phi_i \;\&\; P\equiv(M_1\mid\cdots\mid M_n\mid R)
\end{aligned}$$

Note that if in the last clause we have $\pi\triangleright\varepsilon$, that is $\pi\sim\omega$, then $\models P:\pi$ for any P. The realizability predicate explains the meaning of the functional characters: $\phi_1\times\cdots\times\phi_n\to\phi$ is the property of being a function (this is the content of $M\Downarrow$) whose application to an argument P satisfies ϕ whenever the bag P contains resources $M_1,\ldots,M_n$ that satisfy $\phi_1,\ldots,\phi_n$. One may note that if $\models P:\pi\times\zeta$, then also $\models P:\pi$ and $\models P:\zeta$. Moreover, it is not difficult to see that:

$$\models P:\pi\times\zeta \;\Leftrightarrow\; \exists Q,R.\; P\equiv(Q\mid R) \;\&\; \models Q:\pi \;\&\; \models R:\zeta$$

We extend the realizability predicate to arbitrary terms of Λ^m (that is, not necessarily closed), defining $H\models M:\phi$, where H is a mapping from the set of variables to formulae of Π, such that $H(x)=\omega$ for almost all variables, that is except for a finite set of variables. When we are only interested in the value of H on the set $\{x_1,\ldots,x_k\}$, we write H as $x_1:\pi_1,\ldots,x_k:\pi_k$, where $\pi_i=H(x_i)$. Now we define $H\models M:\ \phi$ as follows, where $\mathsf{fv}(M)=\{x_1,\ldots,x_k\}$:

$$x_1:\pi_1,\ldots,x_k:\pi_k\models M:\phi \;\Leftrightarrow_{\rm def}\; \forall P_1,\ldots,P_k\,\big(\forall i.\; \models P_i:\pi_i \;\Rightarrow\; \models M\langle P_1/x_1\rangle\cdots\langle P_k/x_k\rangle:\phi\big)$$

Obviously this definition coincide with the previous one for M closed. As we shall see in the next lemma, this definition does not depend on the order in which the variables of M are written. For any hypothesis Γ, in the sense of the functionality system, we denote by $\Gamma^\times$ the mapping from variables to Π such that $\Gamma^\times(x)$ is the product of the formulae assigned to x by Γ (if there is none then $\Gamma^\times(x)=\omega$) – this is slightly ambiguous, but it does not matter for our purpose. Now our aim is to show the following *soundness* property:

$$\Gamma\vdash M:\phi \;\Rightarrow\; \Gamma^\times\models M:\phi$$

The following simple lemma will be of crucial use:

LEMMA 3.10. *$H\models N:\phi$ & $N\sqsubseteq_{\mathcal{A}} M \;\Rightarrow\; H\models M:\phi$*

PROOF. We first show that, for M and N closed:

$$\models N:\phi \ \&\ N \sqsubseteq_{\mathcal{A}} M \ \Rightarrow \models M:\phi$$

We proceed by induction on the definition of $\models N:\phi$, that is by induction on ϕ. The case of $\phi = \omega$ is trivial. If $\phi = \pi \rightarrow \psi$ then $N\Downarrow$, hence obviously $M\Downarrow$, and if $\models P:\pi$, then $\models NP:\psi$. Since $NP \sqsubseteq_{\mathcal{A}} MP$ we have $\models MP:\psi$ by induction hypothesis, therefore $\models M:\pi \rightarrow \psi$.

We note that this implies that the definition of $H \models M:\phi$ does not depend on the order of the free variables $x_1,\ldots,x_k$, for if i is a permutation of $1,\ldots,k$ we have, for $P_1,\ldots,P_k$ closed:

$$M\langle P_1/x_1\rangle\cdots\langle P_k/x_k\rangle \asymp M\langle P_{i_1}/x_{i_1}\rangle\cdots\langle P_{i_k}/x_{i_k}\rangle$$

hence also $M\langle P_1/x_1\rangle\cdots\langle P_k/x_k\rangle \simeq_{\mathcal{A}} M\langle P_{i_1}/x_{i_1}\rangle\cdots\langle P_{i_k}/x_{i_k}\rangle$ by the Corollary 2.4. As a matter of fact, we could even have assumed that the set $\{x_1,\ldots,x_k\}$ contains $\mathsf{fv}(M)$ in the defintion of $H \models M:\varphi$.

Now assume that $H \models N:\phi$, and let $\{x_1,\ldots,x_k\} = \mathsf{fv}(N) \cup \mathsf{fv}(M)$. Let $P_1,\ldots,P_k$ be closed terms such that $\models P_i:H(x_i)$ for any i. Then $\models N\langle P_{i_1}/x_{i_1}\rangle\cdots\langle P_{i_n}/x_{i_n}\rangle:\phi$ by definition of $H \models N:\phi$, where $\{x_{i_1},\ldots,x_{i_n}\} = \mathsf{fv}(N)$. By the Corollary 2.4, we have:

$$N\langle P_1/x_1\rangle\cdots\langle P_k/x_k\rangle \simeq_{\mathcal{A}} N\langle P_{i_1}/x_{i_1}\rangle\cdots\langle P_{i_n}/x_{i_n}\rangle$$

Since

$$N \sqsubseteq_{\mathcal{A}} M \ \Rightarrow\ N\langle P_1/x_1\rangle\cdots\langle P_k/x_k\rangle \sqsubseteq_{\mathcal{A}} M\langle P_1/x_1\rangle\cdots\langle P_k/x_k\rangle$$

we get $\models M\langle P_1/x_1\rangle\cdots\langle P_k/x_k\rangle:\phi$ by the previous point. Now if $\mathsf{fv}(M) = \{x_{j_1},\ldots,x_{j_m}\}$ we have, by the Corollary 2.4:

$$M\langle P_1/x_1\rangle\cdots\langle P_k/x_k\rangle \simeq_{\mathcal{A}} M\langle P_{j_1}/x_{j_1}\rangle\cdots\langle P_{j_m}/x_{j_m}\rangle$$

hence $H \models M:\phi$, using the previous point ❑

Let us write $H \sim H'$ whenever $H(x) \sim H'(x)$ for any variable x. Then a next step towards the soundness property is:

LEMMA 3.11. $H \sim H' \ \&\ \phi \sim \psi \ \&\ H \models M:\phi \ \Rightarrow\ H' \models M:\psi$

PROOF. We first show that

$$\phi \sim \psi \ \&\ \models M:\phi \ \Rightarrow \models M:\psi$$

We proceed by induction on the formula ϕ. This is trivial for $\phi = \omega$, since then $\psi = \phi$. Otherwise let $\phi = \pi \rightarrow \phi'$. Then $\psi = \zeta \rightarrow \psi'$ for some ζ and ψ' such that $\zeta \sim \pi$ and $\psi' \sim \phi'$. Let $\models P:\zeta$. Then, from the Remark 3.9 and the induction hypothesis, it should be clear that $\models P:\pi$, therefore $\models MP:\phi'$, hence $\models MP:\psi'$ by induction hypothesis.

Now if $\mathsf{fv}(M) = \{x_1,\ldots,x_k\}$ and $\models P_i:H(x_i)$ for any i, we also have $\models P_i:H'(x_i)$ by the previous point, whence the lemma ❑

To prove the soundness property, we will use the following observation: the decomposition of a formula π into the sequence $\phi_1,\ldots,\phi_n$ corresponds in an exact manner to a decomposition of the typings $\Gamma \vdash P:\pi$. More precisely:

LEMMA 3.12. *Let* $\pi \rhd \phi_1,\ldots,\phi_n$. *Then* $\Gamma \vdash P:\pi$ *if and only if there exist* $M_1,\ldots,M_n \in \Lambda^m$, $R \in \Pi$, *and* $\Gamma_1,\ldots,\Gamma_n$ *such that* $P \equiv (M_1 \mid \cdots \mid M_n \mid R)$, $\Gamma_1,\ldots,\Gamma_n \gg \Gamma$ *and* $\Gamma_i \vdash M_i:\phi_i$ *for any* i *(with a proof shorter than the one of* $\Gamma \vdash P:\pi$*).*

PROOF. The "if" direction should be clear, using the Lemma 3.2. The "only if" direction is proved by a straightforward induction on the inference of the sequent $\Gamma \vdash P : \pi$ ❑

Now we are ready to establish the main result of this section:

PROPOSITION (SOUNDNESS) 3.13. $\Gamma \vdash M : \phi \;\Rightarrow\; \Gamma^\times \models M : \phi$

PROOF. The proof is by induction on the inference of $\Gamma \vdash M : \phi$. Then, as usual, we show that the realizability predicate satisfies the relevant rules of the typing system, that is L1-L4 and L7-L9.

(L1) We have to show $\models M : \varphi \;\&\; P \equiv (M \mid R) \;\Rightarrow \models x\langle P/x\rangle : \phi$. Clearly $x\langle P/x\rangle \to M\langle R/x\rangle$, and $M\langle R/x\rangle \asymp M$ since M is closed, therefore $M \sqsubseteq_{\mathcal{A}} x\langle P/x\rangle$ by the Lemma 2.5 and the Corollary 2.4. Then $\models x\langle P/x\rangle : \phi$ whenever $\models M : \phi$ by the Lemma 3.10.

(L2) Our hypothesis here is $x : \pi, \Gamma^\times \models M : \phi$, where x is not in Γ. Let $x_1, \ldots, x_k$ be the free variables of $\lambda x.M$, and $\Gamma^\times = x_1 : \pi_1, \ldots, x_k : \pi_k$. Let $P_1, \ldots, P_k$ be such that $\models P_i : \pi_i$ for any i. We show that

$$\models (\lambda x.M)\langle P_1/x_1\rangle \cdots \langle P_k/x_k\rangle : \pi \to \phi$$

Since this term is a value, we obviously have $(\lambda x.M)\langle P_1/x_1\rangle \cdots \langle P_k/x_k\rangle\Downarrow$. Now let $\models P : \pi$. Then, by definition of $x : \pi, \Gamma^\times \models M : \phi$, we have, possibly using the Lemma 3.10 if x is not free in M:

$$\models M\langle P/x\rangle\langle P_1/x_1\rangle \cdots \langle P_k/x_k\rangle : \phi$$

Clearly $(\lambda x.M)\langle P_1/x_1\rangle \cdots \langle P_k/x_k\rangle P \to M\langle P/x\rangle\langle P_1/x_1\rangle \cdots \langle P_k/x_k\rangle$, therefore

$$\models (\lambda x.M)\langle P_1/x_1\rangle \cdots \langle P_k/x_k\rangle P : \phi$$

by the Lemma 3.10.

(L3) Assume that $\Gamma^\times \models M : \pi \to \phi$, with $\Delta \vdash P : \pi$, and let $\pi \rhd \phi_1, \ldots, \phi_n$. Then by the Lemma 3.12 there exist $M_1, \ldots, M_n, R$ and $\Delta_1, \ldots, \Delta_n$ such that $P \equiv (M_1|\cdots|M_n|R)$ and $\Delta_i \vdash M_i : \phi_i$ with $\Delta_1, \ldots, \Delta_n \gg \Delta$. By induction hypothesis $\Delta_i^\times \models M_i : \phi_i$ for any i. Let $\{x_1, \ldots, x_k\} = \mathsf{fv}(MP)$ and $P_1, \ldots, P_k$ be such that $\models P_i : (\Gamma, \Delta)^\times(x_i)$ for any i. Since $(\Gamma, \Delta)^\times(x) \sim \Gamma^\times(x) \times \Delta^\times(x)$ we have $P_i \equiv (Q_i \mid R_i)$ where $\models Q_i : \Gamma^\times(x_i)$ and $\models R_i : \Delta^\times(x_i)$. Moreover, for any x there is some ζ such that $\Delta^\times(x) \sim \Delta_1^\times(x) \times \cdots \times \Delta_n^\times(x) \times \zeta$ since $\Delta_1, \ldots, \Delta_n \gg \Delta$. Then $R_i \equiv (R_1^i \mid \cdots \mid R_n^i)$ for some $R_1^i, \ldots, R_n^i$ such that $\models R_j^i : \Delta_j^\times(x_i)$. Therefore the following holds (possibly using the Lemma 3.10):

$$\begin{aligned} \models N : \pi \to \phi \quad &\text{where} \quad N = M\langle Q_1/x_1\rangle \cdots \langle Q_k/x_k\rangle \\ \models N_1 : \phi_1 \quad &\text{where} \quad N_1 = M_1\langle R_1^1/x_1\rangle \cdots \langle R_1^k/x_k\rangle \\ &\vdots \\ \models N_n : \phi_n \quad &\text{where} \quad N_n = M_n\langle R_n^1/x_1\rangle \cdots \langle R_n^k/x_k\rangle \end{aligned}$$

By definition of the realizability predicate, we have $\models (N_1 \mid \cdots \mid N_n) : \pi$, therefore

$$\models N(N_1 \mid \cdots \mid N_n) : \phi$$

By the Lemmas 2.7 and 2.6, we have:

$$N(N_1 \mid \cdots \mid N_n) \sqsubseteq_{\mathcal{A}} (MP)\langle P_1/x_1\rangle \cdots \langle P_k/x_k\rangle$$

hence $\models (MP)\langle P_1/x_1\rangle \cdots \langle P_k/x_k\rangle : \phi$. This shows $(\Gamma, \Delta)^\times \models MP : \phi$.

(L4) Assume that $x:\pi, \Delta^\times \models M:\phi$ where Δ does not contain x and $\Gamma \vdash P:\pi$. Let $\pi \triangleright \phi_1, \ldots, \phi_n$. Then by the Lemma 3.12 there exist $M_1, \ldots, M_n, R$ and $\Gamma_1, \ldots, \Gamma_n$ such that $P \equiv (M_1 | \cdots | M_n | R)$ and $\Gamma_i \vdash M_i : \phi_i$ with $\Gamma_1, \ldots, \Gamma_n \gg \Gamma$. By induction hypothesis $\Gamma_i^\times \models M_i : \phi_i$ for any i. Let $\{x_1, \ldots, x_k\} = \mathsf{fv}(M\langle P/x \rangle)$. We may assume that $x \notin \{x_1, \ldots, x_k\}$. Let $P_1, \ldots, P_k$ be such that $\models P_i : (\Gamma, \Delta)^\times (x_i)$ for any i. As in the previous point, we have $P_i \equiv (Q_i \mid R_1^i \mid \cdots \mid R_n^i)$ where $\models Q_i : \Delta^\times(x_i)$ and $\models R_j^i : \Gamma_j^\times(x_i)$. Therefore the following holds (possibly using the Lemma 3.10):

$$\models N_1 : \phi_1 \quad \text{where} \quad N_1 = M_1 \langle R_1^1/x_1 \rangle \cdots \langle R_1^k/x_k \rangle$$

$$\vdots$$

$$\models N_n : \phi_n \quad \text{where} \quad N_n = M_n \langle R_n^1/x_1 \rangle \cdots \langle R_n^k/x_k \rangle$$

By definition of the realizability predicate, we have $\models (N_1 \mid \cdots \mid N_n) : \pi$, therefore

$$\models M\langle (N_1 \mid \cdots \mid N_n)/x \rangle \langle Q_1/x_1 \rangle \cdots \langle Q_k/x_k \rangle : \phi$$

By the Lemmas 2.7 and 2.6, we have:

$$M\langle (N_1 \mid \cdots \mid N_n)/x \rangle \langle Q_1/x_1 \rangle \cdots \langle Q_k/x_k \rangle \sqsubseteq_{\mathcal{A}} M\langle P/x \rangle \langle P_1/x_1 \rangle \cdots \langle P_k/x_k \rangle$$

hence $\models M\langle P/x \rangle \langle P_1/x_1 \rangle \cdots \langle P_k/x_k \rangle : \phi$. This shows $(\Gamma, \Delta)^\times \models M\langle P/x \rangle : \phi$.

(L7,L9) These cases are trivial (using the Lemma 3.11).

(L8) This case is obvious, since $\Gamma \gg \Delta \Rightarrow \forall x \exists \zeta.\ \Delta^\times(x) \sim \Gamma^\times(x) \times \zeta$, therefore if $\models P : \Delta^\times(x)$ we also have $\models P : \Gamma^\times(x)$ $\square$

Clearly the second half of the computational adequacy property is an obvious consequence of the soundness of the functionality system with respect to the realizability predicate:

COROLLARY 3.14. *For any closed* M:

$$\vdash M : \pi \to \phi \;\Rightarrow\; M \Downarrow$$

This completes the proof of the Adequacy Theorem. As a matter of fact, one can see that, if we denote by $\mathcal{H}(M)$ the set of pairs (H, ϕ) such that $H \models M : \phi$, we have proved the following:

$$M \triangleright N \;\Rightarrow\; \mathcal{F}(N) \subseteq \mathcal{F}(M) \;\Rightarrow\; N \sqsubseteq M \;\Leftrightarrow\; N \sqsubseteq_{\mathcal{A}} M \;\Rightarrow\; \mathcal{H}(N) \subseteq \mathcal{H}(M)$$

It is easy to see that, as usual, the first implication cannot be reversed. For instance, $\mathcal{F}(x\langle \mathbf{1}/x \rangle)$ consists of all the pairs (Γ, ω), as well as $\mathcal{F}(x\langle \Omega/x \rangle)$, but neither of these two terms reduces to the other. Moreover, due to the non-determinism, the reduction is generally strictly decreasing w.r.t. the semantics.

4. Conclusion.

To conclude this paper, let us mention some related work, and briefly examine the possible connections with the λ-calculus with multiplicities that we have proposed. We already mentioned Bounded Linear Logic [13]. Certainly, it would be worth investigating whether our calculus, or more accurately a part of it, provides a syntax for proofs in this logic. In relation to this, we are studying a typed sub-calculus where the types are as our functional characters, except that ω is replaced by a set of propositional variables – possibly subject to universal quantification –, and that we use the exponential $!\phi$.

There are by now some papers, by Abramsky [3], Benton *et al.* [4], Lafont [12,14], Lincoln and Mitchell [15], Mackie [16], Wadler [21], among others, which investigate the possible use of intuitionistic Linear Logic in functional programming. The common expectation is that Linear Logic, as a logic of resources, could help in analysing and solving implementation problems regarding storage management and evaluation strategies. Most of these studies follow the "Curry-Howard paradigm". That is, they develop term calculi representing the proofs of a logical system. This means that every logical rule is recorded in the syntax as a term construction – up to some equivalence of proofs.

Some of the above mentioned authors have noted that this discipline may in fact be quite constraining, especially when one try to use a linear term calculus as a typed functional programming language. For instance Mackie observes in [16] that "algorithms become hidden in a wealth of contractions, weakenings and derelictions". We have taken the opposite approach, ignoring these operations, but obviously we cannot claim that our λ-calculus with multiplicities may be regarded as a programming language, like any other pure – i.e. untyped – calculus. However, since our calculus involves some more refined features than the usual λ-calculus, it is worth investigating whether it can be used as a "machine-oriented" framework for dealing with implementation problems. Moreover, its relationships with various typed calculi, and especially with linear term calculi, deserve to be studied.

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Appendix: α-conversion.

The sets $\mathsf{fv}(T)$ and $\mathsf{bv}(T)$ of free and bound variables of T are defined by:

$$\begin{aligned}
\mathsf{fv}(x) &= \{x\} & \mathsf{bv}(x) &= \emptyset \\
\mathsf{fv}(\lambda x.M) &= \mathsf{fv}(M) - \{x\} & \mathsf{bv}(\lambda x.M) &= \{x\} \cup \mathsf{bv}(M) \\
\mathsf{fv}(MP) &= \mathsf{fv}(M) \cup \mathsf{fv}(P) & \mathsf{bv}(MP) &= \mathsf{bv}(M) \cup \mathsf{bv}(P) \\
\mathsf{fv}(M\langle P/x\rangle) &= (\mathsf{fv}(M) - \{x\}) \cup \mathsf{fv}(P) & \mathsf{bv}(M\langle P/x\rangle) &= \{x\} \cup \mathsf{bv}(M) \cup \mathsf{bv}(P) \\
\mathsf{fv}(\mathbf{1}) &= \emptyset & \mathsf{bv}(\mathbf{1}) &= \emptyset \\
\mathsf{fv}(P \mid Q) &= \mathsf{fv}(P) \cup \mathsf{fv}(Q) & \mathsf{bv}(P \mid Q) &= \mathsf{bv}(P) \cup \mathsf{bv}(Q) \\
\mathsf{fv}(M^{\infty}) &= \mathsf{fv}(M) & \mathsf{bv}(M^{\infty}) &= \mathsf{bv}(M)
\end{aligned}$$

The α-conversion is *not* defined by means of substitution. Instead, we use the operation of *renaming*

x by z in T, denoted $\alpha_x^z(T)$, given by:

$$
\begin{aligned}
\alpha_x^z(y) &= \begin{cases} z & \text{if } y = x \\ y & \text{otherwise} \end{cases} \\
\alpha_x^z(\lambda y.M) &= \begin{cases} \lambda y.M & \text{if } y = x \text{ or } y = z \\ \lambda y.\alpha_x^z(M) & \text{otherwise} \end{cases} \\
\alpha_x^z(MP) &= \alpha_x^z(M)(\alpha_x^z(P)) \\
\alpha_x^z(M\langle P/y\rangle) &= \begin{cases} M\langle \alpha_x^z(P)/y\rangle & \text{if } y = x \text{ or } y = z \\ \alpha_x^z(M)\langle \alpha_x^z(P)/y\rangle & \text{otherwise} \end{cases} \\
\alpha_x^z(\mathbf{1}) &= \mathbf{1} \\
\alpha_x^z(P \mid Q) &= (\alpha_x^z(P) \mid \alpha_x^z(Q)) \\
\alpha_x^z(M^\infty) &= (\alpha_x^z(M))^\infty
\end{aligned}
$$

Then the α-conversion $M =_\alpha N$ is the congruence generated by the following laws:

$$
\begin{aligned}
\lambda x.M &= \lambda z.\alpha_x^z(M) \quad \text{where } z \notin \mathsf{fv}(M) \cup \mathsf{bv}(M) \\
M\langle P/x\rangle &= \alpha_x^z(M)\langle P/z\rangle \quad \text{where } z \notin \mathsf{fv}(M) \cup \mathsf{bv}(M)
\end{aligned}
$$

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